Curſus Mathemati

OR, A

Compleat Courſe

OF THE

MATHEMATICKS.

In Five Volumes.

Vol. I. Contains a ſhort Treatiſe of *Algebra*, and the Elements of *Euclid*.

Vol. II. *Arithmetic* and *Trigonometry*, with correct Tables of Logarithms, Sines and Tangents.

Vol. III. *Geometry* and *Fortification*.

Vol. IV. *Mechanics*, and *Perſpective*.

Vol. V. *Geography* and *Dialling*.

The whole Illuſtrated with near 200 Copper Plates.

Written in *French* by Monſieur *OZANAM*, Profeſſor of the *Mathematicks* at *Paris*.

Now done into Engliſh, *with Additions and Corrections by Several Hands.*

LONDON:
Printed for *John Nicholſon* at the Queen's-Arms in *Little-Britain*, and Sold by *John Morphew* near *Stationers-Hall*. M DCC XII.

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Vol. I.

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The AUTHOR's

PREFACE.

AFTER so many Mathematical Works, that have been already Publish'd, as well in the several Parts, as in a Body, usually call'd a *Course of Mathematics*, in imitation of those that had done the like in other Sciences; I shou'd never have entertain'd the least Thought of increasing the Number, and of composing a New *Cursus*, had not I found those hitherto done were but of little use: Some, because too prolix and voluminous, and by that means, both deterring the less Laborious from medling with them, and distracting the Minds of the most Intent; Others because too concise, by giving them little or no clear Insight into the matter, rather supposing them already acquainted with these things, than making them so; it being almost impossible to be Short, and yet preserve that Clearness which is necessary to instruct Beginners: Lastly, Others are but of small use, because written in foreign Languages, especially *Latin*; and such is the Unhappiness of the

Age, that there are but few young Persons so well acquainted with that Language, as to be able to read Books written in it with any Pleasure, and understand the Terms with Ease.

I flatter'd my self with the hopes of succeeding in my Design, by the great Desire I have of seeing this Art flourish, that has been the distinguishing Character of the most Polite, Ingenious, and Learned Ages, and of the good Dispositions I find in the Minds of the present; For every body courts the Mathematics, especially such of the Nobility and Great Men, as used to distinguish themselves by despising the Learning of the Schools, but are however charm'd with the Beauties of this Science.

The Necessity that Gentlemen are under, that would become considerable in the Art of War, or any great Employment, which cannot subsist without recourse to the Mathematics, makes them leave off several trifling Amusements, and apply themselves to these Sciences; and oftentimes the unexpected Pleasures they meet with, do so surprise and engage them, that they make it ever after as well the delightful as the serious part of their Studies.

I don't promise my Reader any Elegancy of Expression or Stile, which serve only to tickle the Fancy and please the Ear; nor do I invite him to any such Flowery Pleasures and Airy Delights, as the Muses inchant their Admirers withal: But what I propose is solid and substantial, and Pleasures becoming a Reasonable Creature. One may judge of the Genius of a Reader, by the Books he makes choice of, and the value he puts on them: *Achilles* was brought

brought up in the Drefs of the contrary Sex, and fo could not be diftinguifhed; yet no fooner was he prefented, on the one hand with Toys and Trifles, and on the other with Arms, but his Genius, born for great Things, betray'd the fecret of his Education, and it was known by his Choice that he was deftin'd to be a Hero. One may difcover among Children, which of them are born to fomething extraordinary, by their choice of Sports and Amufements; and never was any Child pleas'd with any thing a-kin to the Mathematics, that did not prove confiderable in whatever Employment he was afterwards engaged in.

I fhall fay nothing here of the Ufefulnefs of Mathematics, becaufe I have done it already in my *Mathematical Dictionary*, Printed fome Years ago. And perhaps fome Perfons expect a greater Work than I pretend at prefent to publifh: I know, a Man muft quit all other Studies when he applies himfelf to the Mathematics, or at leaft intermit and fufpend them, till he has acquir'd the Art of Exactnefs and Method, in a word, till he has attain'd the Art of Reafoning well himfelf, and can judge of the Reafoning of another, till he can diftinguifh Truth from Error in all its various Shapes: So that I am afraid of being accus'd of Idlenefs, or Indifference for the Public, in whofe Service I profefs to have been fo long engag'd; I know, generally fpeaking and judging of Things according to their Goodnefs, no Bounds ought to be fet to Mathematical Books, and that one ought to go as far as one can, becaufe 'tis in a Way where a Man can never lofe himfelf, or exhauft the Subject; but I am contrain'd to accommodate my felf to the

Humour

Humour of ſuch as fancy they can be the better by my Labours, becauſe ſhort and eaſy, which otherwiſe would diſhearten them.

Such as ſtudy the pleaſurable part of Life, underſtand the Secret of riſing with an Appetite, without cloying their Taſt; the ſame ought to be obſerv'd by thoſe who apply themſelves to Sciences: Yet I have not in theſe Treatiſes been ſo reſerv'd, but that I have given ſufficient Inſight to any Gentleman that is deſirous to underſtand theſe things, and have diſcovered enough to enable him of himſelf to make what Progreſs he pleaſes, either by reading of Authors, or by his own further Studies and private Reflections.

I have all along endeavour'd to ſpeak with the greateſt Perſpicuity I cou'd, without being confin'd to ſtudied Phraſes or uſeleſs Expreſſions: Nor do I ſuppoſe my Readers at all acquainted with the Art, or any of its Terms, or Ways of Reaſoning, but teach him them, and let no Term, tho' never ſo little out of the way, paſs unexplain'd, that no Difficulty may be left behind.

To inure the Mind to reaſon on Abſtracted Subjects, ſuch as are thoſe of Mathematics, I begin with an *Introduction*, where you'll find a general Idea or Notion of theſe Sciences, the moſt general Terms explain'd in order, together with ſome Problems that may be reſolv'd by Rule and Compaſs, to bring in the Hand of Beginners. And becauſe without *Algebra* a Perſon cannot ſo eaſily diſtinguiſh the Relations of different Species of Quantities, nor reſolve immediately any Problem, much leſs inveſtigate a

Theorem,

Theorem, or find its Demonſtration when the Theorem is known; I thought it proper to inſert in this Introduction *A Compendium of Algebra*, whoſe Name I know ought not to ſcare the Reader, for 'tis only a Method of Reaſoning by the help of the Letters of the Alphabet, repreſenting the Quantities, whoſe Relations are conſider'd; and it is to the Mathematics, the ſame that Logic is to the ordinary Philoſophy, and therefore has been called *Logiſtic*, and is become ſo common amongſt us, becauſe of its engaging Beauty, and vaſt Uſe in all parts of the Mathematics, that even Ladies of the higheſt Quality have been induced to learn it; the Dutcheſs of *E*— has attain'd ſo great a Degree of Perfection, as well in Numbers as Geometry, that Perſons who make the greateſt Figure for Learning have earneſtly ſought for the Honour of her Converſation. An Inſtance ſo illuſtrious ought to baniſh all ſorts of Diffidence, and excite thoſe that love their Eaſe.

And to diſpoſe the Mind, that it may not be taken with falſe Appearances, I have put the *Elements of Euclid* next, that ſerve alſo for a kind of Introduction to the Mathematics, and being well underſtood, will render all the other parts eaſy, as being demonſtrated from theſe Elements: And here you'll find that to become a Mathematician, one muſt draw the Mind from every thing that falls under the Notice of our Senſes, and conſider Quantity perfectly abſtracted; ſo that one muſt begin to reaſon after this abſtracted manner, and accuſtom ones ſelf to Ideas no ways concern'd with Matter, and above all, get a habit of aſſenting to nothing but what is Evident, yield to nothing but what we ſee cannot be otherwiſe; in fine, we muſt

baniſh

banish from Mathematics all that is Doubtful, or but Probable, and entertain nothing but Certainty and Demonstration.

I shall not speak here in particular of the other parts of this *Cursus*, because it would swell the Preface, and deface the Ideas I would impress by the two Introductions; and perhaps make a Person imagine he is thorowly acquainted with them, when he has but just heard them talk'd of. I shall only mention the Parts of the other Volumes, as I have done this; that the Reader, finding at the Beginning of every Volume, particular Considerations upon what is contain'd in it, may enter upon this Study with greater Satisfaction, and if I may so say, Greediness of learning and being acquainted with that, whose Excellency and Usefulness is there laid down.

I shall only say then, that I divide the whole *Course* into five Volumes: The First comprehends *An Introduction to Mathematics*, and *the Elements of Euclid*; the Second, *Arithmetic* and *Trigonometry*, with exact *Tables of Logarithms*, *Sines*, *Tangents*; the Third, *Practical Geometry* and *Fortification*; the Fourth, *Mechanics* and *Perspective*; the Last, *Geography* and *Dialling*.

THE

THE BOOKSELLERS TO THE READER.

THE Study of the Mathematics, in our Nation, being become almost universal, the Usefulness of which is sufficiently recommended by our Author, in his several Prefaces to this Work; and there being in our Language no compleat System yet extant, at least so large and general as this; We, by the Advice and Direction of several of the most eminent in this Science, as well at London *as the Universities of* Great-Britain *and* Ireland, *that this was the most easy, most useful, and the cheapest to the Buyer of any* Course of the Mathematics *yet extant in any Language, resolved to print it in* English; *and having engaged several ingenious Gentlemen, well skill'd in the Parts they undertook, to Translate and Correct the several Volumes, we have with a very great Expence compleated the same; the whole containing Five Volumes,* viz.

The

The First Volume contains an Introduction to the Mathematics, *with the* Elements of Euclid. *The* Introduction *begins with the Definitions of the most general Terms in Mathematics; which are follow'd by a little Treatiſe of* Algebra, *for the better underſtanding of what enſues in the* Courſe, *and ends with many Geometrical Operations, perform'd both upon Paper with Ruler and Compaſſes, and upon the Ground with a Line and Pins. The* Elements of Euclid *comprehend the firſt Six Books, the Eleventh and Twelfth, with their Uſes.*

In the Second Volume we have Arithmetick, *and* Trigonometry *both Rectilineal and Spherical, with* Tables *of Logarithms, Sines and Tangents.* Arithmetic *is divided into Three Parts; the First handles* Whole Numbers, *the Second* Fractions, *and the Third* Rules of Proportion. Trigonometry *has alſo Three Diviſions or Books; the First treats of the* Conſtruction of Tables, *the Second of* Rectilineal, *and the Third of* Spherical Trigonometry: *With* Tables *of* Logarithms, Sines, *and* Tangents. *Theſe* Tables *were carefully Corrected by Mr.* Hodgſon, *Maſter of the Mathematical School at* Chriſt's Hoſpital, London.

The Third Volume comprehends Geometry *and* Fortification. Geometry *is diſtributed into Four Parts, of which the First teaches* Surveying, *or* Meaſuring of Land; *the Second* Longimetry, *or* Meaſuring of Lengths; *the Third* Planimetry, *or* Meaſuring of Surfaces; *the Fourth* Stereometry, *or* Meaſuring of Solids. Fortification *conſiſts of Six Parts; in the First is handled* Regular Fortification, *in the Second* the Conſtruction of Outworks,

works, *in the Third* the different Methods of Fortifying, *in the Fourth* Fortification Irregular, *in the Fifth* Offensive Fortification, *and in the Sixth* Defensive Fortification: *With the Translators* Appendix, *concerning that Method of Fortifying which is truly Mr.* Vauban's.

The Fourth Volume includes the Mechanics, *(to which is added, by way of Notes, what was thought proper out of Dr.* Wallis's *Works,* &c.) *and* Perspective. *In* Mechanics *are Three Books; the First is of* Machines simple and compounded, *the Second of* Statics, *and the Third of* Hydrostatics. Perspective *gives us first the General and Fundamental* Principles of that Science, *and then treats of* Practical Perspective, *of* Scenography, *and of* Shading.

The Fifth Volume consists of Geography *and* Dialling. *Of* Geography *there are two Parts; the First concerning the* Cœlestial Sphere, *and the Second of the* Terrestrial. Gnomonics *or* Dialling *hath Five Chapters; the First contains many* Lemma's, *necessary for the understanding of the Theory and Practice of Dialling, the Second treats of* Horizontal Dials, *the Third of* Vertical Dials, *the Fourth of* Inclined Dials, *and the Fifth of the description of the* Circles of the Sphere *upon all sorts of Dials.*

The First, Second, and the Geometry part of the Third Volume, were look'd over by Mr. Jones, *Professor of Mathematics in* London, *and Fellow of the* Royal Society: *The Fortification, as also the fourth and fifth Volumes were done by Mr.* Desaguliers *of* Hart-Hall *in* Oxford.

This

This Author also writ a large *Mathematical Dictionary*, which is design'd to be Translated into *English*.

There is lately Publish'd, in the same Size as these Volumes.

A
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INTRODUCTION TO THE Mathematics.

MATHEMATICS is a Science which takes under consideration whatever can be measur'd or computed, and because every thing that can be measur'd and computed is a concrete or discrete Quantity, that is to say, continued or discontinued; it follows that the Object of Mathematicks is Quantity or finite Magnitude, such as is capable of increase by Addition or Multiplication, and of decrease by Substraction or Division; and the Quantity that has a sensible extension, call'd *Dimension*, as a Line, Surface, and Solid, and also Time, Motion and Weight, are the Objects of *Geometry*: But the same Quantity that has no sensible extension, such as Number, whose Dimensions are only imaginary, and not to be perceiv'd but by Thought, is the Object of *Arithmetic*.

These two Parts, Arithmetic and Geometry, which constitute what is commonly call'd *Simple Mathematics*, and which *Plato* calls the two Wings of a Mathematician, do mutually help each other, and are the foundation of the other Parts of the Mathematics, commonly call'd *Mix'd Mathematics*, such as *Astronomy*, *Opticks*, *Mechanicks*, &c. which are no other than Physical Knowledge explain'd by the Principles of Arithmetic and Geometry.

Tho' the Mathematics take cognizance only of Quantity, yet they do not consider it absolutely and in it self, but only the relation it may have to another Magnitude of the same kind, by comparing together these two homogeneal Quantities, in order to the finding out some hidden Truth, and afterwards to demonstrate it, by reasons foun-

ded on other Truths, which are naturally known to every body, and are therefore call'd *Common Notions of the Mind*, or *Principles*; of which there are three sorts, *viz. Definitions*, *Axioms*, and *Postulates*.

DEFINITIONS are the explications of such words and terms which concern a Proposition, towards the rendring of it plain and clear, and for avoiding all manner of difficulties and objections, in the demonstration.

AXIOMS, or Maxims, are simple and general Propositions, the knowledge whereof is so evident of it self, that no body can deny them without contradicting their natural sence and reason; so that every rational Man is oblig'd to allow of them, there being no proof more convincing than the natural light of the Mind. As when it is said, that *from one Point to another Point there can but one right Line be drawn*.

POSTULATES are suppositions of certain Practices, the performance whereof is so easy in it self, that no Man of sense and judgment can be ignorant of it, or will contest it. As, *upon a Plane to describe a Circle with a Compass*. They are call'd *Postulates* or *Demands*, because its requir'd and expected that every Man shou'd acknowledge them to be naturally known to all, and so easy that there is no need of any Master to teach them, or to be obliged to demonstrate them.

These three sorts of Principles being granted, the Mathematicians use them for the Demonstration of such Propositions as they advance, which are of two sorts, to wit, the *principal Propositions*, which are either *Problems* or *Theorems*: And the *less principal Propositions*, which are either *Corollaries* or *Lemmas*, which when they have been demonstrated do in their turn conduce to the Proof of other Propositions which depend on them.

A *PROBLEM* is a Question which proposes something to be done, and teaches how to do it, and to construct it by the preceding Principles, touching some Practice commonly necessary to the Demonstration. As, *to find the Centre of a given Circle*. There are several sorts of Problems, some of which will be here explain'd, after having shewn what this word *Given* means.

By this word *Given*, the Mathematicians understand something whose Magnitude, or Position, or Species, or Proportion is known; so that when its Magnitude is known, its said to be *given in Magnitude*; and when its Position is known its said to be *given in Position*: But when its Magnitude and Position are known 'tis said to be *given in Magnitude and Position*. Thus in describing a Circle on a Plane, its

its Centre is given in Position, its Diameter is given in Magnitude, and the Circle is given in Magnitude and Position; and if a Diameter be drawn at pleasure, that Diameter will be given in Magnitude and Position. The Circle can only be given in Magnitude, when that Circle is only imaginary, and when only the Magnitude of its Diameter is known: Lastly when its Species is known, its said to be given in Species; and when the Relation of two Quantities is known, they are then said to be *given in Proportion*, &c.

There are Problems which are call'd *Ordinate* and *Inordinate*, *Determinate* and *Indeterminate*, *Simple*, *Plane*, *Solid*, and *Surfolid*, that is to say, *more than Solid*.

An Ordinate Problem is that which can be done but only one way, As *to make the Circumference of a Circle pass thro' three given Points*; there being but one only Circle, whose circumference can pass thro' three given Points. 5. 4.

An *Inordinate Problem* is that which can be done an infinite number of ways. As *to describe the Circumference of a Circle thro' two given Points*, it being evident that thro' two given Points an infinite variety of Circles may be drawn.

A *Determinate Problem* is that which has but one certain determin'd number of Solutions; as *to divide a given Line into two equal parts*, this Problem having but one Solution; or *to find two whole Numbers, the difference of whose Squares shall be equal to* 48, which has but two Solutions to wit, 8, 4, and 7, 1, for the two Numbers sought for. 5. 1.

An *Indeterminate* or *Local Problem*, is that which is capable of an infinite variety of different Solutions, so that the Point which contributes to the resolution of the Problem, when it is in Geometry, may be taken at pleasure, within a certain extent call'd the *Geometric Place*, which may be a Line, a Plane, or a Solid; and then it it is said that the Problem is a *Place* or *Locus*, which is call'd *Simple Place*, or *Locus ad lineam rectam*, when the Point which resolves the Problem is in a right Line: *Plane Place*, or *Locus ad Circulum*, when that Point is found in the circumference of a Circle: *Solid Place*, when the same Point is found in the circumference of a Conic Section, other than the Circle, as of a *Parabola*, an *Hyperbola*, or of an *Ellipsis*, &c.

A *Simple*, or *Linear Problem*, is such as may be resolv'd Geometrically by the intersection of two right Lines. It is evident that such a Problem is *Ordinate*, because it can have but one Solution, since two right Lines will cut one another but in one Point.

A *Plane Problem* is such as may be resolv'd Geometrically, by the intersection of the circumferences of two Circles, or by the intersection of the circumference of a Circle and a

right Line. It is evident that ſuch a Problem can have but two Solutions becauſe two circumferences of a Circle, or a right Line and the circumference of a Circle, can cut each other but in two Points only.

A *Solid Problem* is that which may be reſolv'd by the interſection of two Conic Sections, other than two Circles. It is evident that ſuch a Problem can have at moſt but four Solutions; becauſe two Conic Sections cannot interſect in more than four Points.

A *Surſolid Problem* is that which cannot be reſolv'd Geometrically, without making uſe of ſome Curve Line of a higher kind than Conic Sections. It is evident that ſuch a Problem is capable of more than four Solutions, becauſe a Curve Line of a higher kind than Conic Sections may be cut by another Curve Line in above four Points.

A Problem that is extremely eaſie and almoſt ſelf-evident, and which ſerves to reſolve more difficult ones, is call'd a *Porima*, from the Greek word *Porimos*, which ſignifies a thing eaſy to be comprehended, and which opens
3. 1. the way to things of a more difficult Nature; *as from a given Line to cut off a leſs given Line.*

A Problem which is poſſible, but which has not ever been reſolv'd, becauſe of its ſeeming difficulty, is call'd an *Apore*; as is now (by ſome) the *Squaring the Circle.* Before *Archimedes* the *Squaring of the Parabola* was an *Apore.*

By this word *Quadrature* or *Squaring* is meant, in the Mathematics, the manner of reducing into a right lined Figure a *Curve lined Figure*, that is to ſay, a Figure bounded by Curve Lines, becauſe all right lined Figures may be eaſily reduc'd into Squares. Thus the *ſquaring the Parabola* is the way of finding a right lined Figure equal to a Parabola; and the *Squaring the Circle* is the manner of deſcribing a right lined Figure equal to a given Circle.

5. 1. A *THEOREM* is a determinate Propoſition touching the Nature and Properties of a thing, ſhewing how to find out an hidden Truth, and to deduce it from its proper Principles. Of which ſort is this Propoſition, which lays down, that *when the two Sides of a Triangle are equal, the two Angles at the Baſe are alſo equal.*

A general Theorem which is diſcover'd in any *Locus* found, is call'd a *Poriſma*; ſo that when, either by the ancient or modern Analyſis, the conſtruction of any local Problem is found out, and a general Theorem drawn from the conſtruction of that *Locus*, ſuch a Theorem is call'd a Poriſma. A Poriſma therefore is no other than a Corollary deliver'd like a Theorem that is diſcover'd in a *Locus*, with its conſtruction and demonſtration, ſerving, ſays *Pappus*, for the

the conſtruction of the moſt general and difficult Problems.

The word *Poriſma* comes from the Greek *Poriſo* which according to *Proclus* ſignifies to eſtabliſh and conclude from what has been done and demonſtrated, which made him define a Poriſma to be a Theorem drawn occaſionally from ther Theorem done and demonſtrated.

A *COROLLARY* is a neceſſary and evident Truth, that is to ſay a conſequence evidently drawn from what has been done or demonſtrated. As if from a preceding Theorem, we learnt that *the two Angles of a Triangle are equal, when* 5. 1.
the two oppoſite Sides are equal, it is concluded that *the three Angles of an equilateral Triangle are equal*.

A *LEMMA* is a Propoſition put where it is to ſerve for the Demonſtration of a Theorem, or Reſolution of a Problem; it is commonly put before the Demonſtration of the Theorem to the end its Demonſtration ſhou'd be leſs perplex'd; or before the reſolution of a Problem, to render it the ſhorter, and therefore 'tis that *Euclid* in his Elements teaches how *to draw an equilateral Triangle*, before he ſhews how from 1. 1.
a *Point given to draw a right Line equal to one given*, and 2. 1.
that he always demonſtrates a Theorem before its inverſe, which in another Place we have call'd a *Reciprocal Theorem*.

Among the leſs principal Propoſitions, we may likewiſe put the *Scholium* which ſhall be explain'd after we have ſhewn what Demonſtration means, together with its different kinds.

DEMONSTRATION is one or many Syllogiſms, or ſucceſſive reaſonings drawn one from another, which clearly and invincibly demonſtrate a Propoſition, that is to ſay, which convince the mind of the truth or falſity, of the poſſibility or impoſſibility of a Propoſition; and without Demonſtration there is always reaſon to doubt of any Propoſition, unleſs it be a Principle, becauſe it frequently happens that a Propoſition is falſe, when it ſeems true to the Senſes, and even to the Mind, which is often impos'd upon by the Senſes, when it has not ſufficiently examin'd the thing.

Theſe Reaſonings are founded on the three ſorts of Principles before mention'd, in properly applying them to each other, that is to ſay, in applying one truth to another truth, and from theſe two truths concluding a third, and thus by continuing to deduce truths from truths, by a proper and orderly uſe not only of Definitions, Axioms and Demands, already granted, but likewiſe of Theorems, Problems, Lemmas, and Corollaries, till we arrive at the laſt Truth, call'd the *Concluſion*, becauſe it concludes and fully

and perfectly convinces the Mind of what was to be Demonstrated.

Besides the Conclusion, there belongs to a Demonstration the *Hypothesis*, which is a supposition of the things known or given in the Proposition to be demonstrated or constructed; as also the *Preparation*, which is a construction made beforehand by drawing some Lines either real or imaginary, to perform the Demonstration with the greater ease, and more readily conduct the Mind to the knowledge of the truth propos'd to be demonstrated.

There are several sorts of Demonstrations of which the two most considerable are those which we call *Positive*, or *Affirmative*, or *Direct*; and *Negative*, or *Impossible*, or *Indirect*.

A *Positive*, *Affirmative*, or *Direct Demonstration* is that which by affirmative and evident Prepositions, drawn directly from each other, does at last discover the truth sought for, and concludes with what it pretended to demonstrate, so that it forces the Reason to consent to such
1. 1. a truth. Of which sort is that in *Prop.* 1. B. 1. of *Euclid*'s Elements, and many others.

A *Negative*, *Impossible*, or *Indirect Demonstration* is that which demonstrates a truth by some absurdity which necessarily follows, if the proposition advanc'd and contested shou'd not be true. *Euclid* therefore to demonstrate, that *a Triangle*
6. 1. *which has two Angles equal has also two Sides equal*, shews that the part wou'd be equal to its whole, if one of those two Sides were greater than the other, from whence he concludes they must be equal.

Each of these two ways of Demonstration equally convince the Mind, and oblige it to consent to the Truth demonstrated, but do not equally enlighten it; for 'tis certain that the *Direct* is more satisfactory and clear than the *Indirect*. Wherefore the latter is not to be us'd but when it can't be avoided. *Euclid* indeed has made use of Indirect Demonstrations in many Propositions, but we shall endeavour to render them Direct as much as possible.

A *SCHOLIUM* is a Remark made on the Construction of a Problem, or on the Demonstration of a Theorem. As if after having found the Resolution of a Problem, it be remark'd that in several Cases the Resolution might have been done a shorter way by Compendiums drawn from the general Resolution: Or if after having demonstrated a Theorem by *Synthesis*, it be remark'd that the Demonstration might likewise have been perform'd by *Analysis*. But now it concerns us to explain what is *Synthesis*, and what *Analysis*.

SYN-

SYNTHESIS or *Composition* is the Art of finding out the truth of a Proposition, by Consequences regularly drawn from establish'd Principles, or by Propositions which demonstrate each other, beginning at the most simple, and proceeding on to the more compound, until the last be attain'd, which finishes the conviction of the Mind as to the truth sought for, and obliges it to assent thereto: So that whosoever shall consider with attention the consequence of all these propositions, shall be invincibly convinc'd of it, and shall no longer be able to refuse his consent to this last truth, of which before he was in doubt, or absolutely ignorant of.

ANALYSIS, or *Resolution* is the Art of discovering the truth of a Proposition by a way contrary to that of Composition, to wit, by supposing the Proposition such as it is, and by examining what follows from this Proposition, untill one arrives at some clear truth, of which what has been suppos'd is a necessary consequence, to conclude from it the truth of the Proposition, by making use of Composition by a retrograde order, namely by taking up its reasonings where the other ended. You have an example of Synthesis and Analysis in *Theor.* 3. *Part* 3. *Chap.* 1. *of Geometry.*

Analysis when it is us'd in pure *Geometry*, as the Ancients did, consists more in the judgment and in the application of the Mind, than in particular Rules: But at present it is made use of in *Algebra*, which is a literal Arithmetic by the means whereof hidden truths are more easily and methodically found out. I shall give you what M. *Prestet* says of it in his *New Elements of Mathematics*.

" Never cou'd the Synthesis of Geometricians have ar-
" riv'd to so high a pitch as it has done in this Age,
" had not the Analysis of the Moderns supported it, and
" brought to light an infinite number of fine discoveries
" unknown to the most learned among the Ancients. It is
" indeed impossible to argue by any other way more in-
" geniously, methodically, profoundly or learnedly, and
" more compendiously. Its expressions by Letters are al-
" together simple and familiar, and the Mind can be sup-
" ply'd with nothing of so great help in the discoveries
" of truth, because they lessen its labour, and dextrously
" save its application, they fix it and render it attentive
" upon the Object of its enquiries, they commodiously
" point out all the parts of them, they support the ima-
" gination, they renew and spare the Memory as much as
" possible, in a word, they rule and perfectly guide the
" Mind, and yet so little do they divide or employ it by

" the Senses, that they leave it an entire liberty to exert
" all its vigour and activity in its search after truth. So
" that nothing can escape its penetration; and the justness
" or clearness of its reasonings does commonly discover
" the shortest way to the truths it seeks after, or the
" Mediums that are wanting to arrive at it, when they are
" beyond its reach.

These and many other reasons have made me of opinion that since Algebra is at present more esteem'd and more cultivated than ever, it wou'd not be amiss, before any other thing, for the sake of beginners, to add a Compendium of this noble Science, at least as much as we have need of in *Euclid*'s Elements, and elsewhere, to soften the Demonstrations which seem more difficult by any other way than by the Analysis of Geometricians; and to add lastly some Geometrical Problems, which we shall resolve by Rule and Compass upon Paper, and with a Stick and Chord or Chain upon the Ground, by simple and easy Practices, without any Demonstrations, to bring their hand in who never us'd such Instruments, and to dispose them the better to understand *Euclid*'s Elements, and the other Treatises which ought to follow them.

A COMPENDIUM OF Algebra.

ALGEBRA is a Science by means whereof we endeavour to resolve any possible Problem in the Mathematics, which is done by the means of a sort of literal Arithmetic, which for that reason has been call'd *Specious Algebra*, because its reasonings are all done by the species or forms of things, namely by the Letters of the Alphabet, which are extremely helpful to the memory and imagination of those who apply themselves to this noble Science: For without that, all those things which serve to discover the truth sought for, must be retain'd in the Mind, which requires a strong Imagination, and cannot be done without great labour to the memory.

These Letters represent each in particular either *Lines* or *Numbers*, according as the Problem is propos'd touching *Geometry*, or *Arithmetic*; and being join'd together, they represent *Planes*, *Solids*, and higher Powers according to their Number; for if there are two Letters together, as *ab*, they represent a *Rectangle*, whose two dimensions are represented by the Letters *a*, *b*, namely one side by the Letter *a*, and the other side by the other Letter *b*, so that being multiplied together, they produce the *Plane ab*. And if there are two like Letters as *aa*, this Plane *aa*, will be a *Square*, whose side is *a*, which is call'd *Square Root*.

But if there are three Letters together as *abc*, they will represent a *Solid*, namely a *Rectangular Parallelepipedon*, whose three

three dimensions will be expreſs'd by the Letters *a*, *b*, *c*, to wit, the length by the Letter *a*, the breadth by *b*, and the height or depth by the laſt letter *c*, to the end that theſe three Letters being multiplied together they may produce the ſolid *abc*. So that if theſe three Letters are the ſame as *aaa*, this Solid *aaa*, will repreſent a *Cube*, whoſe ſide is *a*, which is call'd *Cube Root*.

Laſtly, if there are more than three Letters together, they will repreſent a higher Power, of as many dimenſions as there are Letters: and ſuch Powers are call'd *Imaginary*, becauſe in nature there is no ſenſible Quantity known, which has more than three dimenſions. This Power, or imaginary Quantities call'd *Plano-Plane* or a *Power of four dimenſions*, when it is expreſs'd by four Letters, as *abcd*, and when theſe four Letters are the ſame as *aaaa*, this Plano-Plane *aaaa*, is call'd *Square-Squar'd*, whoſe ſide is *a*, which is call'd *Square-Squar'd Root*.

This ſame Power is call'd *Plane-Solid*, when it is repreſented by five Letters: and when they are the ſame, as *aaaaa*, it is call'd *Surſolid*, whoſe ſide is *a*, which is call'd *Surſolid Root*.

Thus you ſee that theſe Powers go on encreaſing by a continual addition of Letters, which is equivalent to a continual Multiplication: And when they are compos'd of equal Letters, they are call'd *Regulars*, and *Vieta* calls them *Gradual Quantities*, becauſe they encreaſe by a degree conformable to the number of their Letters. Thus *aa*, is a *Power of the ſecond Degree*, becauſe it has two Letters; and *aaa*, is a *Power of the third Degree*, becauſe it has three Letters, and ſo on.

From whence it follows that the *Root*, or the common *Side a*, of all thoſe Powers, is a *Power of the firſt Degree*.

But as by augmenting theſe gradual Quantities by a continual addition of the ſame Letter, the Number of the Letters may become ſo great, that it will be hard to reckon them, and even to write them upon Paper, in ſuch caſe it will ſuffice only to write the Root, that is to ſay, only one Letter, and to annex to it towards the right hand a Figure expreſſing the number of the Letters, of which the Power is compos'd, and this number is call'd *Exponent* of the ſame Power, and ſhews the Number of its Dimenſions, it is commonly written a little higher than the Letters, ſo as not to confound them with the other Numbers, when there are any, or when there is any other Letter which follows after at the right hand. Thus to expreſs a *Surſolid*, or a *Power of the fifth Degree*, that is to ſay, of five Dimenſions, whoſe ſide or Root is *a*, inſtead of repreſenting

ting it by these five Letters $aaaaa$, you may represent it thus, a^5. To express likewise the *Cube* of a, you may write thus a^3, and to express the *Square-squar'd* of it, you must write thus a^4. So of others.

It is easily seen by what we just now said, that the gradual Quantities, or the Powers of any Root, as a, are

$$a^1, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, \&c.$$

this natural Series, and that they are in a Geometrical Progression, while their Exponents are in an Arithmetical Progression, because the Powers encrease by a continual Multiplication of one and the same Root, and their Exponents augment by a continual addition of that of the same Root, which is 1, tho' not always written, because it is understood, for it is evident that a is equivalent to a^1.

Thus putting for a, what number you will, for example 2, then a^2 will be 4, a^3, will be 8, and the other Powers will be such as you see here, which shew that the Powers, or gradual Quantities, 2, 4, 8, &c. are in a Geometrical

$$a^1, a^2, a^3, a^4, a^5, a^6, a^7, a^8, \&c.$$
$$2, 4, 8, 16, 32, 64, 128, 256, \&c.$$

Progression, and that their Exponents 1, 2, 3, &c. are in an Arithmetical Progression. Which is the cause that these Exponents may be consider'd as the Logarithms of their Powers. From whence it follows that the Exponent of a Power which is produc'd by the Multiplication of two other Powers, is equal to the Sum of the Exponents of those Powers. Thus the *Sursolid* 32, hath 5, for its Exponent, namely the Sum of the Exponents 1, 4, of the Powers 2, 16, which produce it, or of the Exponents 2, 3, of the Powers 4, 8, which produce it.

Thus you see that there is a great difference between $3a$, and a^3, because a^3, signifies the Cube of the Root a, and $3a$ represents the triple of that Root: So that if a be equal to 2, its Cube a^3 is equal to 8, and its Triple $3a$, is only equal to 6, in like manner $3a^4$, expresses the triple of the Square-squar'd of the Root a, so that if a be equal to 2, the Plano-Plane $3a^4$ is equal to 48. So of others.

CHAP. I.

Of Monomes, or Simple Quantities.

WHAT we call *Monomes* is a literal Quantity which subsists alone, that is, such as is not accompanied with any

any other Quantity connected by this Character $+$, which signifies *more*, or by this $-$, which signifies *less*.

PROBLEM I.

To add one Quantity to another.

As homogeneal Quantities do not affect the heterogeneal ones, that is to say, that one Quantity cannot augment another Quantity of a different kind, when it is added to it, nor diminish it, when it is substracted from it; it follows, that those which are to be added together, ought to be homogeneal, that is to say, of the same kind; and when they are of the same kind, let their Coefficients, be added together, and the same Letters, and the same Exponents retain'd, and when they are of divers kinds they may be added by the Sign $+$, because *more*, as well as *less*, does not make different kinds. This Addition will be easily comprehended by the following Examples, where you

$2a$	$2a^3$	$2abb$	$2a$	$2aab$
$4a$	$4a^3$	$4abb$	$3b$	$3abb$
$3a$	$8a^3$	$10abb$	$2a+3b$	$4a^3$
$9a$	$14a^3$	$16abb$		$2aab+3abb+4a^3$

may see that by the Addition of several Quantities of the same kind, there one only Quantity is found, which consequently is also a Monome; and by the addition of several Quantities of different kinds, a *Polynome* is form'd, which we will call Binome, when it is compos'd of two Monomes, which are call'd *Terms* as $2a+3b$; and *Trinome*, when it is compos'd of three Monomes or Terms, as $2aab+3abb+4a^3$ &c.

PROBLEM II.

To Substract one Quantity from another.

Substraction likewise supposes Quantities to be homogeneal; for it is evident that a Plane cannot be diminished by the Substraction of a Line, because a Plane is compos'd of an infinite number of Lines, nor a Solid by the Substraction of a Line, or Plane, because a Solid is compos'd of an infinite number of Lines, and also of Planes.

As we have said that the Sign *less* does not make different kinds, a Quantity may be substracted from another Quan-

Quantity greater and of the same kind, by taking its Coefficients from those of the greater, and by retaining the same Letters, and their Exponents: and from another Quantity greater and of a different kind, by writing it after the greater towards the right hand, and by connecting them with the Sign —, which belongs to the Quantity that is to be substracted, which in this case is call'd *Negative Quantity*, altho' it be positive in it self, being negative only in respect to that from which it is to be substracted. See the following Examples.

$6a$	$8aa$	$12abb$	$3a$	$2a^3$
$2a$	$3aa$	$4abb$	$2b$	$2aab$
$4a$	$5aa$	$8abb$	$3a-2b$	$2a^3-2aab$

It often happens that a greater Quantity is required to be substracted from a less, which being absolutely impossible, the less must be substracted from the greater, as was just now taught, and the Sign — must be prefix'd to the remainder, to shew that, that remainder proceeds from the substraction of a greater Quantity from a less, and consequently is a negative Quantity. Thus substracting $5a$ from $3a$, the remainder will be $-2a$, and substracting $10bb$ from $3bb$, the remainder will be $-7bb$, and so for others.

To represent the excess of one Quantity above another Quantity of a different kind, without knowing which is the greater; as if we cannot tell to which of these two Quantities the Sign — ought to be attributed, they must be join'd by this .... which signifies *Difference*. Thus the difference of these two Quantities $2a$, $3b$, is $2a \ldots 3b$, or $3b \ldots 2a$, and the difference of these two $2a^3$, $4abb$, is $2a^3 \ldots 4abb$, or $4abb \ldots 2a^3$.

PROBLEM III.

To multiply one Quantity by another.

Multiplication does not any more than Division require the Quantities to be homogeneal, for nothing hinders but a Plane may be multiplied by a Line, and it will become a Solid; or a Solid by a Line, and it will become a Plano-Plane. Thus you see that the Multiplication of Quantities changes the kind, and elevates it, except when it is made by a Number, in which case the same kind remains.

First to multiply a literal Quantity by a number, multiply the Coefficient of that literal Quantity by that number,

ber, and retain the same Letters and their Exponents. Thus to multiply this literal Quantity $3aabb$ by 4, you must multiply 3 by 4, and you will have $12aabb$ for the Product.

But to multiply one literal Quantity by another, the Coefficients must be multiplied together, and the Exponents added, if the Letters are the same in each of the Factors, otherwise write down the Letters one after another with their Exponents, and prefix the Product of their Coefficients, as in the following Examples, where you may observe that the Exponent of a Square is double, that of its Root, the

$2a$	$2aa$	$3a$	$9aa$	$18aabc$
$3b$	$4aa$	$3a$	$3a$	$4aacd$
$6ab$	$8a^4$	$9aa$	$27a^3$	$72a^4bccd$

Exponent of a Cube is triple that of its Root, and that the Exponent of a Square-squar'd is quadruple that of its Root.

PROBLEM IV.

To Divide one Quantity by another.

DIvision which *Vieta* calls *Application*, does not as we have already said require the Quantities to be homogeneral, for oftentimes a Quantity of a *higher Power*, that is to say of a higher kind, or which has more Dimensions, is divided by one of a *lower kind*, or by one of a fewer dimensions, as a Plane by a Line, and then a Line is produced: Or a Solid by a Line, and then the Quotient is a Plane. So of the rest. But a continued Quantity cannot be divided by another higher continued Quantity, Geometrically speaking, because that is against the nature of the Quantity, but you may divide a Quantity by a Quantity of the same kind, and then the Quotient is absolutely a Number, generally speaking.

First, if the Divisor be a Number, divide the Coefficient of the Dividend, by that Number, and retain the same Letters and their Exponents: Thus, dividing $8abb$, by 4, the Quotient will be $2abb$, and dividing $32a^3$ by 8, the Quotient will be $4a^3$.

But if the Divisor consist of one or more Letters, and that these same Letters are found in the Dividend, which I suppose rais'd higher than the Divisor; then divide the Coefficients of the Dividend by those of the Divisor, and substract the Exponents of the Letters of the Divisor, from the Exponents of the Letters of the Dividend, and the Letters

Letters which remain without an Exponent, will vanish, and the others will remain in the Quotient, and will be Integers, if the Divisor has not Letters different from those of the Dividend, or if all the Exponents of the Divisor be substracted from the like Exponents of the Dividend, otherwise those different Letters must be plac'd beneath, or else the difference of the Exponent with the same Letters, found by substracting the lesser from the greater, as you see in the last of the following Examples.

$6a^3b^3$ $(2aabb$	$9a^3b^4$ $(\frac{9}{2}aab^3$	$8aab^6$ $(\frac{2ab^5}{cc}$
$3ab$	$2ab$	$4abc$
0	0	0
$12a^5b^4$ $(4a^4bb$	$13a^4b^3$ $(\frac{13}{6}a^3b$	$16aab^3cc$ $(\frac{2aab}{cc}$
$3abb$	$6abb$	$8bbc^4$
0	0	0

PROBLEM V.

To extract the Root of a given Quantity.

WE have remark'd in Multiplication, that the Exponent of a Square is double that of its Root, that the Exponent of a Cube is triple that of its Root, and so on. Wherefore to extract the Square Root of a given Quantity, you must take the Square Root of its Coefficient, and the half of its Exponent, and to extract the Cube Root of it, you must take the Cube Root of its Coefficient, and the third of its Exponent. Thus the Square Root of $64a^6b^6$, is $8a^3b^3$, and its Cube Root is $4aabb$, which has likewise its Square Root $2ab$. So of others.

A Power which has neither + nor — prefix'd, is accounted *affirmative*, that is to say, prefix'd by a +, and then it will always have the Root sought, provided it has a Number which has such a Root prefix'd, and that its Exponent be divisible exactly by that of the same Root, to wit by 2, for the Square Root, by 3, for the Cubic Root, and so on. Thus the Square Root of $4a^8b^8$ is $2a^4b^4$, and the Cube Root of a^6b^6, is is $aabb$, the Coefficient being understood in the Root as well as in the Power; for it is evident that a^6b^6, is equivalent to $1a^6b^6$, and its Cube Root $aabb$ equivalent to $1aabb$.

If

If the Power whose Root is to be extracted be *negative*, that is to say has $-$ prefix'd, it will never have such a Root, altho' it has the Quality which we mention'd, unless the Exponent of the Root sought be an odd Number, and then the Root will be also negative. Thus the Cubic Root of $-8a^3b^3$, is $-2ab$, and the Sursolid Root of $-32a^{10}b^5$, is $-2aab$. But $-4aabb$ has no Square Root, but such as is call'd *Imaginary*, which is express'd thus, $\sqrt{-4aabb}$, the Mark $\surd$ signifying Root.

When a given Quantity has no Root, the Character $\surd$ is prefix'd with the Exponent of the Root, placed above that radical Sign. Thus the Cube Root of $12a^3b^3$, is express'd in this manner, $\sqrt[3]{12a^3b^3}$, and the Square Root of $24aabb$, is writ thus, $\sqrt[2]{24aabb}$, or plainly thus, $\sqrt{24aabb}$, the Exponent 2 being understood, which is neglected to be written, when you wou'd represent a Square Root. And such Roots are commonly call'd *Irrational Quantities*.

These Roots or irrational Quantities may be express'd, more plainly, when the Power is divisible by another Power which has the Root sought for, to wit, by writing the radical Sign $\surd$ between the Root of this other Power and the Quotient. Thus for the Cube Root of $12a^3b^3$, instead of $\sqrt[3]{12a^3b^3}$, write $ab\sqrt[3]{12}$, because the Power $12a^3b^3$, is divisible by this a^3b^3, which its Cubic Root ab, and the Quotient is 12. In like manner to represent the Square Root of this Power, $6aabb$, instead of writing thus, $\sqrt{6aabb}$, you may write thus $ab\surd 6$, because the given Power $6aabb$, is divisible by this $aabb$, the Square Root whereof is ab, and the Quotient is 6.

CHAP.

CHAP. II.

Of Polynomes, or Compound Quantities.

YOU have ſeen in the preceding Chapter, that by the Addition and Subſtraction of ſeveral Quantities of different kinds, a Polynome is formed, the *Terms* of which, that is to ſay, the Monomes which compoſe it, may be differently *affected*, that is to ſay, Affirmative or Negative, according as they have been added or ſubſtracted: Now leſt the diſtinction of + and −, which are call'd *Signs*, ſhou'd cauſe ſome difficulty, before you come to the Practice, we ſhall here add the following Theorems.

THEOREM I.

The Sum of two Quantities affected alike, is of the ſame affection.

THat is to ſay, that if any two Quantities are *Affirmative*, or have + prefix'd, their Sum will be Affirmative; and if they are *Negative*, their Sum will be alſo Negative. For it is evident that the Sum $a+b$, of the two Quantities a, b, or $+a$, $+b$, which are *affected alike*, that is to ſay, have the ſame Sign prefix'd, which ſhew that they are both Affirmative, is Affirmative, becauſe if they were negative, that is, $-a-b$, each of theſe two Quantities would be alſo negative, which is contrary to the Suppoſition. It is evident alſo that the Sum $-a-b$, of the two negative Quantities $-a$, $-b$, is negative, becauſe if it was affirmative, ſo that it were $a+b$, each of thoſe two Quantities would be alſo affirmative, which is alſo contrary to the ſuppoſition. Thus it is ſeen that + added to + makes +, and that − added to − makes −.

THEOREM II.

The Sum of two unequal Quantities differently affected, is of the ſame affection with the greater, and is equal to their Difference.

FOR ſince they are differently affected, by the ſuppoſition, the one ought to be affirmative and the other negative, and their Sum being compos'd of a negative Quantity and an affirmative one, ſhews that the negative Quantity ought to be ſubſtracted from the affirmative one, becauſe Negation is a mark of Subſtraction. Wherefore if the Negative is leſs than the Affirmative, it may be ſubſtracted from the Affirmative, and then there will remain a part of the Affirmative, ſo that the Difference will be Affirmative, and of the ſame Affection with the greater. *Which is one of the two things which was to be Demonſtrated.*

But if the negative Quantity be greater than the affirmative, as the negative cannot be ſubſtracted from the affirmative, which is ſuppos'd leſs, you muſt ſubſtract the leſs from the greater, that is to ſay the affirmatrve from the negative, and there will remain a part of the negative, ſo that the Difference will be negative, and conſequently of the ſame Affection with the greater. *Which remain'd to be Demonſtrated.*

Thus the Sum of $-2a$ and $+5a$, is $+3a$; and the Sum of $+2a$ and of $-5a$, is $-3a$. From whence it follows that the Sum of two equal Quantities differently affected is o, or nothing.

THEOREM III.

To ſubſtract one Quantity from another, is the ſame thing as to add to that other Quantity the former, affected by a contrary Sign.

THus, for example, if you would ſubſtract $+2a$ from $+5a$, that is, if to $+5a$ you would add $-2a$; becauſe the taking away of an Affirmative is ſubſtituting a Negative, and the Sum $+3a$ will be the Remainder.

It is the ſame if you would ſubſtract $-2a$ from $-5a$, that is, if to $-5a$ you would add $+2a$; becauſe the taking away of a Negation is ſubſtituting an Affirmation, and the Sum $-3a$ will be the Remainder.

But

But if you would substract $+2a$ from $-5a$, that is, if to $-5a$ you would add $-2a$, the Sum $-7a$ will be the Remainder: And if you'd substract $-2a$ from $+5a$, that is, if to $+5a$ you would add $+2a$, the Sum $+7a$ will be the Remainder.

THEOREM IV.

The Product of two Quantities affected alike is affirmative, and the Product of two Quantities differently affected is negative.

It is evident that if two Quantities are affirmative, their Product will be also affirmative; because in multiplying an affirmative Quantity by another affirmative Quantity, you add it as many times as there are Units in that other Quantity; for Affirmation is a mark of Addition: and as this Addition is made by an affirmative Quantity, the Sum which is the Product will be also affirmative.

It is also evident, that if the two Quantities which are multiplied are negative, their Product will be still affirmative: because in multiplying one negative Quantity, by another negative Quantity, you substract it as many times as there are Units in that other negative Quantity; for Negation is a mark of Substraction, and as this Substraction is made by a negative Quantity, the Negation is destroyed, and consequently the Affirmation is restored; so that the Remainder which is the Product, will be affirmative.

Lastly it is evident, that if one of these two Quantities be negative, and the other affirmative, their Product will be negative: because in multiplying the negative by the affirmative, you add it as often as there are Units in the affirmative, and as this is an Addition of negative Quantities, the Sum or the Product will be negative. Furthermore, in multiplying an affirmative by a negative Quantity, you substract it as often as there are Units in that negative Quantity, and as this is a Substraction of affirmative Quantities, by destroying the Affirmation you substitute a Negation, so that the Remainder or Product is negative.

Thus you see that $+$ multiplied by $+$ makes $+$, that $-$ multiplied by $-$ makes $+$; and that $-$ multiplied by $+$, or $+$ by $-$ makes $-$.

THEOREM V.

The Quotient of two Quantities alike affected is affirmative; and the Quotient of two Quantities differently affected is negative.

THis Theorem is evident by the preceding one, because if the Quotient of two Quantities alike affected were not affirmative, as in multiplying the Quotient by the Divisor, you'd have the Quantity which was divided, the Product would not be of the same Affection with that Quantity. The same Inconvenience would happen if the Quotient of two Quantities differently affected were not negative. Therefore, &c.

PROBLEM I.

Addition of Polynomes or Compound Quantities.

HAving written down the Polynomes one under another in order, as in Vulgar Arithmetic, so that Quantities of the same kind, when there are any, may answer each other respectively; add Quantities of the same kind, as was taught in the preceding Chapter, and write those of different kinds below the line, each with its one Sign, as in the following Examples, where we have followed the Rules of + and −, which have been taught in *Theor.* 1, 2.

$$\begin{array}{l} 3a^3b + 3a^4 - 6aabb - 7ab^3 \\ 7a^3b - 5a^4 + 3aabc - 4bbcc \\ \hline 10a^3b - 2a^4 - 6aabb + 3aabc - 7ab^3 - 4bbcc \end{array}$$

$$\begin{array}{cc} a^3 & -\ 3aab \\ 4a^3 & +\ 3aab \\ \hline 5a^3 & 0 \end{array}$$

PRO-

PROBLEM II.

Substraction of Polynomes or Compound Quantities.

TO substract one Polynome from another Polynome, you must by *Theor.* 3. change the Signs of the Polynome to be substracted, that is to say, + must be made —, and — must be made +, then add that Polynome so changed, to that from which you would substract, by the Precepts of the preceding Problem, and the Sum will be, by *Theor.* 3. the Remainder required, as in the following Examples.

$$\begin{array}{l} 6aabb - 3a^3b + 4abbc \\ 2aabb - 5a^3b + 6abbc \\ \hline 4aabb + 2a^3b - 2abbc \end{array} \qquad \begin{array}{r} 8ab + 2bb + 4cc \\ 2ab - 3bb - 2cc + 3cd \\ \hline 6ab + 5bb + 6cc - 3cd \end{array}$$

PROBLEM III.

Multiplication of Polynomes.

HAving put the Multiplicator under the Polynome to be multiplied, as in Vulgar Arithmetic, multiply the superior Polynome by each Term of the inferior, according to the Precepts of the preceding Chapter, observing the Rules of + and —, which have been taught in *Theor.* 4. then add all the Products together, as in the following Examples; where the last save one shews that the Square of the Binome $a + b$, is the Trinome $aa + 2ab + bb$, which may serve as a Rule for the Extraction of the Square Root.

$$\begin{array}{r} 2a + 4b \\ 2a + 2b \\ \hline + 4ab + 8bb \\ 4aa + 8ab \\ \hline 4aa + 12ab + 8bb \end{array} \qquad \begin{array}{r} 2a + 3b \\ 2a - 3b \\ \hline - 6ab - 9bb \\ 4aa + 6ab \\ \hline 4aa \quad 0 \quad - 9bb \end{array}$$

$$\begin{array}{r} 2aa - 2bb \\ 2aa - 2bb \\ \hline -4aabb + 4b^4 \\ 4a^4 - 4aabb \qquad\quad \\ \hline 4a^4 - 8aabb + 4b^4 \end{array} \qquad\qquad \begin{array}{r} aa + 2ab + bb \\ 2ab - bb \\ \hline -aabb - 2ab^3 - b^4 \\ 2a^3b + 4aabb + 2ab^3 \qquad\quad \\ \hline 2a^3b + 3aabb \quad 0 - b^4 \end{array}$$

$$\begin{array}{rl} a + b & \\ a + b & \text{Side} \\ \hline + ab + bb & \\ aa + ab \qquad & \\ \hline aa + 2ab + bb & \text{Square} \\ a + b & \text{Side} \\ \hline + aab + 2abb + b^3 & \\ a^3 + 2aab + abb \qquad & \\ \hline a^3 + 3aab + 3abb + b^3 & \text{Cube} \end{array}$$

as well in literal Quantities as in numbers: And the last shews that the Cube of the same Binome $a+b$, is this Quadrinome $a^3 + 3aab + 3abb + b^3$, which may likewise serve as a Rule for the Extraction of the Cube Root, as well in literal Quantities as in numbers.

PROBLEM IV.

Division of Polynomes.

FIrst, to divide a Polynome by a Monome (or a single Quantity,) each Term of the Polynome ought to be divided one after another by that Monome, according to the Precepts of the foregoing Chapter, and the Quotients put to the Right-hand, as in Common Arithmetic, with the Signs + and −, according to the Rule in *Theor.* 5. as in the following Examples, which may be understood at sight.

$$
\begin{array}{l}
2a)\ 8a^6 + 4a^2b^3c^2 - 3a^2b^4\ (4a^5 + 2ab^3c^2 - \tfrac{3}{2}ab^4 \\
8a^6 + 4a^2b^3c^2 - 3a^2b^4 \\
\hline
0 \qquad 0 \qquad 0
\end{array}
$$

$$
\begin{array}{l}
-2a)\ 9a^5 - 12a^3b^2 - 4b^2c^3 \\
9a^5 - 12a^3b^2 - 4b^2c^3\ (-\tfrac{9}{2}a^4 + 6a^2b^2 + \dfrac{2bbc^3}{a} \\
\hline
0 \qquad 0 \qquad 0
\end{array}
$$

But if the Divisor be a Polynome, let the Terms be placed as in Common Division, and as in the two preceding Examples, then begin to divide at the highest Power with respect to the Letters that are in the Divisor, and finish the rest as in Common Arithmetic, and as in the following Examples.

$$
\begin{array}{l}
2a + 2b)\ 4aa + 12ab + 8bb\ (2a + 4b \\
4aa + 4ab \\
\hline
0 \quad 8ab + 8bb \\
8ab + 8bb \\
\hline
0 \qquad 0
\end{array}
$$

$$
\begin{array}{l}
3a - b)\ 9aa - 6ab + b^2\ (3a - b \\
9aa - 3ab + b^2 \\
\hline
0 \quad -3ab + b^2 \\
-3ab + b^2 \\
\hline
0 \qquad 0
\end{array}
$$

$$
\begin{array}{l}
2bc - cx)\ 4ab^2c - 2abcx + 6b^2c^2 + bc^2x - 2c^2x^2\ (2ab + 3bc + 2cx \\
4ab^2c - 2abcx + 6b^2c^2 - 3bc^2x \\
\hline
0 \qquad 0 \qquad 0 \quad + 4bc^2x - 2c^2x^2 \\
+ 4bc^2x - 2c^2x^2 \\
\hline
0 \qquad 0
\end{array}
$$

If after having multiply'd the Divisor by the Quotient, the Product cannot be substracted for want of Quantities of the same kind, set down this Product below, changing its Sign of + or − into its contrary, because of Substraction, then proceed to divide till all the Terms be brought down, as in the following Examples.

$$\begin{array}{l}
2a+3b)\ 4aa-9bb\ (2a-3b \\
\qquad\quad\ 4aa+6ab \\
\hline
\qquad\quad\ 0-6ab-9bb \\
\qquad\qquad\ -6ab-9bb \\
\hline
\qquad\qquad\quad 0 \qquad 0
\end{array}$$

$$\begin{array}{l}
aa+2ab+bb)\ 2a^3b+3a^2b^2-b^4\ (2ab-bb \\
\qquad\qquad\qquad 2a^3b+4a^2b^2+2ab^3 \\
\hline
\qquad\qquad\qquad\quad 0 \quad -a^2b^2-2ab^3-b^4 \\
\qquad\qquad\qquad\qquad\ -a^2b^2-2ab^3-b^4 \\
\hline
\qquad\qquad\qquad\qquad\qquad 0 \quad 0 \quad 0
\end{array}$$

$$\begin{array}{l}
a+b)\ a^3+b^3\ (aa-ab+b^2 \\
\qquad\ \ a^3+a^2b \\
\hline
\qquad\ \ 0-a^2b+b^3 \\
\qquad\quad\ -a^2b-ab^2 \\
\hline
\qquad\qquad 0+ab^2+b^3 \\
\qquad\qquad\ \ +ab^2+b^3 \\
\hline
\qquad\qquad\quad 0 \qquad 0
\end{array}$$

$$\begin{array}{l}
a-b)\ a^3-b^3\ (a^2+ab+b^2 \\
\qquad\ \ a^3-a^2b \\
\hline
\qquad\ \ 0+a^2b-b^3 \\
\qquad\quad\ +a^2b-ab^2 \\
\hline
\qquad\qquad 0+ab^2-b^3 \\
\qquad\qquad\ \ +ab^2-b^3 \\
\hline
\qquad\qquad\quad 0 \qquad 0
\end{array}$$

If at the end of a Diviſion there remains any thing, or that you cannot divide becauſe of ſome different Letter in the Diviſor and Dividend, make a Fraction of theſe two Polynomes, by putting the Diviſor under the Polynome to be divided, with a line between. Thus dividing, $aa+bb$ by $a+b$, the Quotient will be $\frac{aa+bb}{a+b}$, and dividing a^3+b^3 by $a-b$, the Quotient will be $\frac{a^3+b^3}{a-b}$. So of others.

PRO-

PROBLEM V.

To Extract the Root of a Polynome.

WE have said in Multiplication, that the Trinome $aa+2ab+b^2$, whose Square Root is $a+b$, serves as a Rule to extract the Square Root by: And to shew you how, let us seek the Square Root as if we did not know it, which must be done after this manner.

Forasmuch as the Terms aa and bb are Squares, you may begin at which you will of these two; if you begin at aa, put its Square Root a towards the Right-hand, like a Quotient, for the first Letter of the Root which is

$$\begin{array}{l} aa+2ab+b^2\ (a+b \\ a \\ \hline 0+2ab+b^2 \\ \quad\ \ 2a+b \\ \hline \quad\ \ 0\quad\ 0 \end{array}$$

sought for, and also under the Square aa, so that by multiplying a by a its Square may be had, which being substracted from the Trinome $aa+2ab+b^2$, put the Remainder $2ab+b^2$ under the Line; and since in this Remainder there is $2a$ in the Term $2ab$, it is evident that you must divide $2ab$ by $2a$, which is the double of the first found Letter a, and you will have $+b$ for the second Term of the Root sought: wherefore this second Letter b must be put on the Right-hand, with its Sign $+$ after the first a, and also under its Square b^2, which is the last Term of the Remainder $2ab+b^2$, so that under this Remainder $2ab+b^2$, you will have $2a+b$ for the Divisor, and since there remains nothing after having multiplied and substracted, as the Rule of Division prescribes, one may conclude that the Square Root of the proposed Trinome $aa+2ab+bb$ is precisely $a+b$.

In the same manner the square Root of any other Power is extracted as in the following Examples.

$$a^4 + 4a^3b + 6aabb + 4ab^3 + b^4 \; (aa + 2ab + bb$$
$$aa$$
$$\overline{\qquad\qquad\qquad\qquad\qquad}$$
$$0 + 4a^3b + aabb$$
$$2aa + 2ab$$
$$\overline{\qquad\qquad\qquad}$$
$$0 \quad 2aabb + 4ab^3 + b^4$$
$$2aa + 4ab + bb$$
$$\overline{\qquad\qquad\qquad}$$
$$0 \quad 0 \quad 0$$

$$9a^4 - 36a^3b + 72ab^3 + 36b^4 \; (3aa - 6ab - 6bb$$
$$3aa$$
$$\overline{\qquad\qquad\qquad\qquad\qquad}$$
$$-36a^3b + 72ab^3$$
$$6aa - 6ab$$
$$\overline{\qquad\qquad\qquad}$$
$$-36aabb + 72ab^3 + 36b^4$$
$$6aa - 12ab - 6bb$$
$$\overline{\qquad\qquad\qquad}$$
$$0 \quad 0 \quad 0$$

If in the second Example the square Root had been begun to be extracted at the last Term $36b^4$, this square Root would have been found to be $6bb + 6ab - 3aa$, whose Signs $+$ and $-$ are contrary to those of the first found Root $3aa - 6ab - 6bb$, which shews that a Polynome has always two square Roots, as well as a Monome, and every other Power; and generally speaking, a Quantity has as many Roots, as the Exponent of that Root has Units.

We have also said in the same place, that is to say, in *Prob.* 3. that the Quadrinome $a^3 + 3aab + 3abb + b^3$, whose Cube Root is $a + b$, serves for a Rule to extract the Cube Root by; and to shew how, we will seek for this Cube Root as if we knew it not, thus:

Since the Terms a^3 and b^3 are Cubics, begin at which you will of those two; if you begin by a^3, put its Cube Root a towards the Right-hand, as before, for the first Letter of the Root sought, the Cube of which a^3, ought to be substracted from the proposed Polynome, and the Remainder $3aab + 3abb + b^3$, must be written under the Line, and divided by $3aa$, the triple of the square of the first found Letter a, because in the first Term $3aab$, of the Remainder $3aab + 3abb + b^3$, this triple is found, and the Quotient $+b$ put towards the Right-hand, as before, for the second

$$\begin{array}{l}
a^3 + 3aab + 3abb + b^3 \ (a + b \\
a^3 \\
\hline
0 + 3aab + 3abb + b^3 \\
\quad 3aa \\
\hline
\quad 0 \ + 3abb + b^3 \\
\quad\quad - 3abb \\
\hline
\quad\quad 0 \ + b^3 \\
\quad\quad\quad - b^3 \\
\hline
\quad\quad\quad 0
\end{array}$$

second Letter of the Root sought, and the Remainder of the Division will be $3abb + b^3$, from which you must substract $3abb$ and b^3, to wit, triple the Solid under the first found Figure a, and the Square bb of the second b, and the Cube of the same second; and as nothing remains, it shews that the Cube Root of the proposed Polynome $a^3 + 3aab + 3abb + b^3$ is exactly $a + b$.

If the proposed Polynome has not such a Root as is required, you must express that Root by this Mark $\surd$, which put towards the Left-hand of the Polynome, with a Line over the same Polynome, shewing that the Character $\surd$ does affect the whole Polynome. So to express the Square Root of this Binome $aabb + aacc$, you must write thus, $\sqrt{aabb + aacc}$, or thus, $a\sqrt{bb + cc}$, because the Binome $aabb + aacc$ is divisible by the Square aa, whose Side is a, and the Quotient is $bb + cc$. In like manner to express the Cube Root of this Binome $a^3b^3 + a^3c^3$, you must write $\sqrt[3]{a^3b^3 + a^3c^3}$, or thus, $a\sqrt[3]{b^3 + c^3}$, because the Binome $a^3b^3 + a^3c^3$ is divisible by the Cube a^3, whose Side is a, and the Quotient is $b^3 + c^3$. So of others.

CHAP.

CHAP. III.

Of EQUATIONS.

AN EQUATION is a Comparison which is made between different Quantities, which we would bring to an Equality, and for this purpose are commonly separated by this Character $=$, which signifies *Equal*.

These two Quantities are called *Sides or Members of the Equation*; they are commonly compos'd of several Monomes or Terms, of which all those that are on one and the same side of the Equation, that is to say, in one and the same Member, are consider'd together as one Quantity.

An Equation always follows the Analytical Resolution of a Problem, and at least contains one unknown Quantity, which are commonly express'd by the last Letters of the Alphabet x, y, z, the known Quantities are express'd indifferently by the other Letters. Thus in the Equation $xx + 2ax = bc$, the unknown Quantity is x, which is the reason that the two Terms xx, $2ax$, where it is found, are called *unknown Terms*, which are commonly placed on the same side: and the Term bc where it is not found, is called the *known Term*, as also the last Term, which commonly makes the other side of the Equation, in order to compare it with the unknown; therefore it is that *Vieta* calls it *Homogeneum Comparationis*, tho' others call it the *Absolutely known Quantity*.

Among all the Terms of an Equation, the *first* is that wherein you have the highest Power of the unknown Quantity; the *second*, that wherein the same Quantity is one degree less; the *third*, that wherein the same Quantity is two degrees less than the highest Power, and so on to the *last* Term: As in this Equation, $x^3 + axx - bbx = acc$, the first Term is x^3, the second axx, the third bbx, and the last acc.

Tho' amongst all the Terms of an Equation the degree of the unknown Quantity is not equally decreas'd, by reason of some Term wanting, which often happens, yet that hinders not but that the Term where the unknown Quantity is, for instance, abated two Degrees below the first, may be called the third, tho' it be the second in order. Thus in the following Equation, $x^4 + aaxx + b^3x = c^4$, where the second Term is wanting, the first Term is x^4, the third is $aaxx$, the fourth is b^3x, and the last is c^4.

All

All the Terms of an Equation ought to be homogeneal, at least in Geometrical Problems; and those wherein the unknown Quantity happens to be equally raised, or those wherein it is not found, ought to be accounted as one Term only, as in this Equation, $xx + ax + bx = ad + bd$, the first Term is xx, the second is $ax + bx$, and the last is $ad + bd$.

An Equation is said to be of as many Dimensions as the unknown Quantity in the first Term, that is to say, it is call'd an Equation of *two Dimensions*, or *Quadratic*, if the Square of the unknown Letter be found in the first Term; or of *three Dimensions*, or *Cubic*, if the Cube of the same unknown Quantity happens in the first Term, *&c.* Thus the following Equation $x^3 - abx = aab$, is of three Dimensions, or Cubic, because the Cube of the unknown Quantity x is found in the first Term. And when in the Equation there is only one Term unknown, it is call'd a *Pure Equation*; as $x^3 = abb$, or $xx = ab$, &c.

The unknown Quantity of an Equation may have as many different or equal Values, as the Equation has Dimensions: Thus in this Equation of two Dimensions, $xx + 2x = 15$, there are two Roots, namely $+ 3$, which being affirmative, is call'd a *true Root*; and $- 5$, which is a negative Root, and by *Des Cartes* call'd a false Root; that is to say, x may be supposed $= + 3$, or $= - 5$. This has need of a Demonstration, but we shall say no more of it in this place. See *Des Cartes*'s Geometry.

When one of the Roots of an Equation which depends on some Problem is found, that Problem is resolved. But to find this Root, the Equation shou'd be so reduced, that the first Term be multiplied by no other Quantity than Unity, which is always understood, tho' not mention'd, or at least by another Quantity, which has a Root whose Exponent is equal to the number of Dimensions of the Equation.

Further, all unknown Terms ought to be on one and the same side of the Æquation, which for that reason is called the *unknown Side or Member*, and also *first Side or Member*, because it is commonly written first on the Left-hand, and the known Terms on the other side, which is commonly placed on the Right-hand after this Character $=$.

To conclude, the Equation ought to be brought down as much as possible, that is, it ought to be so reduc'd, that the unknown Quantity be brought to the lowest Degree possible, for the more easy finding out the Roots. This Reduction may be perform'd by means of the following Problems.

PRO-

PROBLEM I.

To Reduce an Equation by Antithesis.

ANTITHESIS is made use of to transpose the Terms of an Equation from one side to another, when they are not disposed as they should be, which is commonly such that the first Term be put first in order, and immediately follow'd by the second, if it is not wanting; and that in like manner the second be follow'd by the third, and so on to the last Term.

If the Term to be transpos'd from one side to the other be affirmative, it must be substracted from each side, and if negative it must be added, for by this means the Terms are transpos'd, and the Equation still preserv'd free from any confusion, according to the Axiom which tells us, that *if to two equal Quantities equal ones are added or substracted, the Sums or Differences will be equal.*

As in this Equation $x^3 - 3axx = b^3 - bbx + 2axx$, if you put all the unknown Terms on the left hand, that is to say, on the first side, you must add to each side the Term bbx, which is negative, and substract the Term $2axx$, which is affirmative; and the propos'd Equation $x^3 - 3axx = b^3 - bbx + 2axx$, will be chang'd into this, $x^3 - 5axx + bbx = b^3$.

From this general Rule the following Compendium may be drawn, for to transpose any Term given from one side to another; *Strike out the Term to be transposed, and put it on the other Side with a contrary Sign.* Thus the following Equation $x^4 + aabb - aacc = aaxx - c^3x$, may be changed into this, $x^4 - aaxx + c^3x = aacc - aabb$, or into this, $x^4 - aaxx + c^3x + aabb - aacc = 0$.

PROBLEM II.

To Reduce an Equation by Parabolism.

IT is not sufficient that by the means of *Antithesis* all the unknown Terms of an Equation may be brought to one side, to find their Roots; but the first Term must likewise have a Root conformable to the number of Dimensions of the Equation, namely a Square Root if the Equation be of two Dimensions, a Cube Root if the Equation be of three Dimensions, and so on.

To

To this end, there needs no more, but to let the Coefficient of the first Term be Unity, if it be found multiplied by any other Quantity than Unity, which may be done by *Parabolism*, to wit, by dividing each side of the Equation by the known Quantity which multiplies the first Term, and this will by no means destroy the Equation, by the Axiom which teaches us, that *if equal Quantities are divided by one and the same Quantity, the Quotients will be equal.*

As if in this Equation, $axx + 2abx = bcc$, each Side be divided by a, the Coefficient of the first Term axx, you'll have this other Equation $xx + 2bx = \frac{bcc}{a}$: and in like manner if this other Equation $abx^3 + aabbx = c^3dd$, be divided by the known Quantity ab, which multiplies the first Term abx^3, you will have this other Equation, $x^3 + abx = \frac{c^3dd}{ab}$. So of others.

PROBLEM III.

To Reduce an Equation by Isomeria.

ISOMERIA is us'd to clear an Equation from Fractions, which are always troublesome in Calculation. To do this, you must first multiply the propos'd Equation by the Denominator of the Fraction to be destroy'd, and the Equation produced must in like manner be multiplied by the Denominator of another Fraction, if there be one, and so on.

Let us propose this Equation, $\frac{x^3}{4} + axx - \frac{bccx}{a} = abb$, and multiply it by the Denominator 4 of the Fraction $\frac{1}{4}x^3$, and we shall have this Equation, $x^3 + 4axx - \frac{4bccx}{a} = 4abb$, which being multiply'd by the Denominator a of the other Fraction $\frac{4bccx}{a}$, you will have this last Equation without Fractions, $ax^3 + 4aaxx - 4bccx = 4aabb$.

For a shorter Method, multiply the propos'd Equation $\frac{x^3}{4} + axx - \frac{bccx}{a} = abb$, by the Product $4a$ of the Denominators 4 and a of the two Fractions $\frac{x^3}{4}$, $\frac{bccx}{a}$, and you'll have this other Equation without Fractions, $ax^3 + 4aaxx - 4bccx = 4aabb$.

PRO-

PROBLEM IV.

To Reduce an Equation by Hypobibasm.

HYPOBIBASM is an equal abatement of all the degrees of the unknown Quantity of an Equation, when that unknown Quantity is found in all the Terms: and this abatement is made by taking away the least Power of the unknown Quantity, so that the Dimensions of the Equation is by this means lessen'd. Thus the Equation $x^4 + 2ax^3 = bbxx$, which seems to be of four Dimensions, is reduc'd to this $xx + 2ax = bb$, which is but of two Dimensions: and this Equation $x^4 - aax = c^3x$, which seems likewise to have four Dimensions, is reduc'd to this, $x^3aax = c^3$, which has but three Dimensions. So for the rest.

PROBLEM V.

To Reduce an Equation by Multiplication.

FOR the avoiding of Fractions which commonly proceed from Division, when you wou'd that the first Term of an Equation shou'd have a Root, whose Exponent is equal to the number of its Dimensions; then multiply each Member of the Equation by the Coefficient of the first Term, if the Equation be Quadratic; or by the Square of that Coefficient, if the the Equation be Cubic, and so on. This Operation will not in the least destroy the Equation, by the Axiom which teaches us, that *if equal Quantities be multiply'd by one and the same Quantity, the Products will be equal*; and the Equation proposed will be found reduced to another, whose first Term will have such a Root as was required.

Thus to make a Square of the first Term of this Quadratic $axx + bcx = bbd$, multiply it by the Coefficient a of the first Term axx, and you'll have this other Equation $aaxx + abcx = abbd$, whose first Term $aaxx$ has ax for its Square Root. Likewise that the first Term of this Cubic Equation $ax^3 + bcxx - bbcx = c^4$, may be a Cube, multiply it by the Square aa of the Coefficient a of the first Term ax^3, and you will have this other Equation, $a^3x^3 + aabcxx - aabbcx = aac^4$, whose first Term a^3x^3 has ax for its Cube Root. The like of others.

Sometimes

Sometimes you may make use of Compendiums, for it signifies little by what Quantity you multiply the given Equation, provided the Root of the first Term be such as was required. So in this Equation $aax^3 + abcxx = abc^3$, if you would have the first Term become a Cube, it will be sufficient to multiply the Equation by a, for then you'll have this other Equation $a^3x^3 + aabcxx = aabc^3$, whose first Term a^3x^3 is a Cube.

PROBLEM VI.

To Reduce an Equation by Division.

By Division we may also make the first Term of an Equation have a Root conformable to the number of its Dimensions, namely by reducing it by *Parabolism*, as you have seen in *Prob.* 2. without any further repetition.

It may also sometimes be of use to bring down an Equation, namely when that Equation is divisible by a Binome, compos'd of the unknown Quantity and of an aliquot part of the last Term, which in this case will be one of the Roots of the given Equation, to wit, the affirmative Root if in the Divisor it be negative, and the negative Root if it be affirmative. This supposes that the Equation should in such a manner be reduc'd by *Antithesis*, that all its Terms shou'd be on one and the same side, and o on the other side.

Thus, by dividing this Equation of three Dimensions $x^3 - bxx - axx - 2abx - aab = 0$, by $x - a$, you'll have this Equation of two Dimensions $xx + ax + bx - ab = 0$. We have several different ways to find such a Divisor, which we shall explain upon some other occasion.

PROBLEM VII.

To Reduce an Equation by Extraction of Roots.

An Equation may also be brought down by extracting the Square or Cubic Root of each side, when that is possible. To this end, it is sufficient that the unknown side of the Equation has the Root which is requir'd; for it signifies little whether the known Side, that is to say, the last Term, has any such Root or no, because being known, it may be always express'd Geometrically, by finding some mean Proportionals when it is irrational.

Thus, to bring down this Equation, $xx + 2ax + aa = bc$, the Square Root of each Side must be extracted, and then you will have this Equation of a lower degree, $x + a = \sqrt{bc}$

or $x + a = d$, by suppoſing the Quantity d a mean Proportional between the two b, c, in which caſe $bc = dd$.

In like manner, to bring down the following Equation, $x^3 + 3axx + 3aax + a^3 = b^3$, you muſt extract the Cube Root of each ſide, and you'll have this Equation $x + a = b$, in which you will find by Antitheſis $x = b - a$, for one of the three Roots of the given Equation.

If the unknown ſide of the given Equation has not ſuch a Root as is required, ſo that ſomething remains, and that this remainder be known, you muſt add to each ſide if it be negative, or you muſt ſubſtract if affirmative, and then the Equation may be brought down.

As in this Equation, $x^3 + 6axx + 12aax = abb$, by extracting the Cube Root of the unknown ſide $x^3 + 6axx + 12aax$, there remains $- 8a^3$. Wherefore you muſt add $8a^3$ to each ſide of the Equation, and you will have this other Equation, $x^3 + 6axx + 12aax + 8a^3 = abb + 8a^3$, where extracting the Cube Root of each ſide, you have this Equation brought lower $x + a = \sqrt[3]{abb + 8a^3}$.

Furthermore, becauſe by extracting the ſquare Root of the unknown ſide of this Equation, $x^4 - 2ax^3 + aaxx - 2bbxx + 2abbx = 3b^4$, there remains $- b^4$, you muſt add b^4 to each ſide, and you have this other Equation, $x^4 - 2ax^3 + aaxx - 2bbxx + 2abbx + b^4 = 4b^4$; where extracting the ſquare Root of each ſide, you will have this other Equation more brought down, $xx \ldots xx \ldots bb = 2bb$.

When all the Terms of the Equation are on one ſide only, ſo that there is 0 on the other, it is not neceſſary that the Remainder after the Extraction of the Root ſought for, ſhou'd be known, and it ſuffices that it hath ſuch a Root, becauſe being added to each ſide of the Equation, you will have another Equation which may be brought lower.

As in this Equation, $9aabb - 24aabx + 12aaxx - 18abxx + 12ax^3 = 0$, by extracting the ſquare Root of the unknown ſide, there remains $- 4aaxx - 12ax^3 - 9x^4$, which ſhews that $4aaxx + 12ax^3 + 9x^4$, which has a ſquare Root, muſt be added to each ſide, then you have this other Equation, $9aabb - 24aabx + 16aaxx - 18abxx + 24ax^3 + 9x^4 = 4aaxx + 12ax^3 + 9x^4$, whoſe ſquare Root gives this Equation in lower Terms, $3ab \ldots 4ax \ldots 3xx = 2ax + 3xx$.

This Method may be applied to all Quadratic Equations, as in this, $xx - 4ax = bb$, where by extracting the ſquare Root of the unknown ſide $xx - 4ax$, there remains $- 4aa$; for if $4aa$ be added to each ſide, you will have this

this other Equation, $xx - 4ax + 4aa = bb + 4aa$, whose square Root gives this Equation in lower Terms, $x - 2a = \sqrt{bb + 4aa}$, in which you will find by *Antithesis*, $x = 2a + \sqrt{bb + 4aa}$, for the affirmative Root, or $x = 2a - \sqrt{bb + 4aa}$, for the negative Root of the Equation propos'd, $xx - 4ax = bb$.

Since the remains after the extraction of the square Root is always equal to the Square of the Coefficient of the second Term, an Equation of two Dimensions may be brought lower by this Compendium.

Add the Square of half the Coefficient of the second Term to each Side of the Equation, and you'll have another Equation, which may be brought lower by extracting the Square Root.

Let us propose for example this Quadratic Equation, $xx + 6ax = bb$, and add to each side thereof the square $9aa$ of the half $3a$ of the Coefficient $6a$ of the second Term $6ax$, and you will have this other Equation $xx + 6ax + 9aa = bb + 9aa$, where by extracting the square Root of each side, this lower Equation, $x + 3a = \sqrt{bb + 9aa}$ is had.

This Method may be also apply'd to higher Equations, where there are but two unknown Terms, such that the greatest Exponent of the unknown Quantity is double the least, because such an Equation is derivative from an Equation of two Dimensions when it is a Bi-quadratic; a *Derivative Equation* being in general where the Exponents of the unknown Letter have one common Measure greater than Unity; as $x^4 + abxx = bbcc$, or $x^6 - 2aabx^3 = aab^3c$.

Thus you have a general Rule, to find by Calculation, the Roots of an Equation of two Dimensions, and of its Derivatives, which is sufficient at present. If you would have any more, see the general Method which we have taught in our *Treatise of Curves of the first kind*, to find the Roots of Equations of two and of three Dimensions, by Calculation.

The same Method may be also apply'd to Equations of three and of four Dimensions, which may be brought lower by taking away the second Term, the practice of which is a great deal longer and more laborious, than by the Extraction of Roots, as we could shew in some Examples, if our Design were not to be brief.

Wherefore to finish this little *Treatise of Algebra*, till we give a more ample one of it, we shall only add here some Arithmetical Questions, to shew you the application of the Rules which we have taught concerning the Reduction of

Equations, and to put you into a Method to resolve several others, in imitation of those that we are going to give, in which you'll find it necessary to exercise your self, if you have a design to make any Progress in it.

A COLLECTION OF SOME Arithmetical Questions, RESOLV'D BY *The New Analysis.*

THE Reasonings we are oblig'd to make, in order to arrive to the resolution of a Question, being express'd on Paper by the Letters of the Alphabet, it is evident that those Letters represent the known Quantities in the Question, and likewise those that are sought for, which, as we have already said, are commonly express'd by the last Letters of the Alphabet, x, y, z, &c.

The known and unknown Quantities, which serve to resolve the Question, being assum'd in Letters, the Question is suppos'd as resolved; and from this Supposition are drawn as many Equations as can be, according to the conditions of the Question, by comparing those Quantities together, to find their relations, which is done by Adding them together, or by Substracting them one from the other, or by Multiplying them, or by Dividing them by one and the same Quantity, as occasion requires, until an Equation be found, which being resolv'd by the Problems of the preceding Chapter, you will at last find the Value of the unknown Letter; which must be substituted in the first Equations found, when there are several unknown Quantities, to find in one of these Equations the Value of another

ther unknown Quantity, which must be likewise substituted until you come to an Equation where there is but one unknown Quantity, in order to be able to discover it there, and so on for the rest, as you see in the following Questions, which will illustrate to you what I have said.

QUESTION I.

Three Persons found 120 Crowns, about which they differed, and each took what he could. The first said, that if besides the Money he had taken, he had 2 Crowns, he shou'd have enough to buy a certain Horse which was to be sold: The second said that he wanted 4 Crowns to be able to buy the Horse: And the third said he wanted 6. The Question is, What the Price of the Horse was, and how many Crowns each Person had?

TO resolve this Question, put the Letter x for the Price of the Horse, and then the first Person's Money will be $x - 2$, the second Person's Money will be $x - 4$, and the third Person's Money will $x - 6$; And because all this Money, namely $3x - 12$, ought to make 120 Crowns, by supposition, you will have this Equation $3x - 12 = 120$, or adding 12 to each side, then $3x = 132$, and dividing by 3, you will have 44 for the Value of the Horse. Thus the value of the Horse is 44 Crowns, from which subtracting 2 Crowns, because of $x - 2$, you will have 42 Crowns for the first Person's Money; and if from the same 44 Crowns you substract 4 Crowns, because of $x - 4$, you will have 40 Crowns for the second's Money; and lastly, if from the same 44 Crowns you substract 6 Crowns, because of $x - 6$, you will have 38 Crowns for the third Person's Money. Now it is evident that the Sum of these three Numbers 42, 40, 38, which are the Sums of Money each of the three Persons had got, is 120: And thus the Question is resolv'd.

SCHOLIUM.

To the end that you may not be oblig'd to renew the Analysis, when the Numbers which are given in the Question are varied, put Letters for those Numbers, as a for 120, b for 2, c for 4, d for 6, and then the Money of the first will be $x - b$, that of the second $x - c$, and that of the third $x - d$; and as all this Money, which is equivalent to $3x - b - c - d$, ought to be equal to the given number

ber a, you will have this Equation, $3x - b - c - d = a$, which being reduc'd by *Antithesis* and by *Parabolism*, will give $x = \frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c + \frac{1}{3}d$, for the general Resolution of the Question; understanding by the *General Resolution*, that which is made in Letters, because it serves generally to resolve the Question for any given numbers whatever. Thus in this Question, whatever value be given to the four Letters a, b, c, d, the Question will be found resolv'd, without which there would be need of a new Analysis, namely by restoring to the Letters a, b, c, d, their suppos'd Values. This is easily conceiv'd, and we shall not amuse our selves hereafter, so as to say any more of it.

QUESTION II.

A Person going into a Church, gives 5 Pence to a Beggar, and in going out finds that the Remainder of his Money was doubled: He goes into another Church, where he gives 100 Pence to the first Beggar he meets, then he had but two Crowns or 120 Pence left. The Question is, how much Money he had when he went into the first Church.

If x be put for the Money that he had when he went into the first Church, there will remain $x - 5$ in going out, because it is suppos'd that he gave 5 Pence to the Poor: And as it is also suppos'd that this remainder was doubled, he had $2x - 10$ in going into the second Church, where having again given 100 Pence to the Poor, if from $2x - 10$, 100 be substracted, the remainder will be $2x - 110$, which by supposition ought to be equal to 120. So that you will have this Equation, $2x - 110 = 120$, to which adding 110, you will have $2x = 230$, and dividing by 2, you will have $x = 115$ for the Resolution of the Question.

QUESTION III.

A Merchant is to pay 250 Pounds at 4 Payments, viz. at the second Payment 1 l. more than at the first, at the third Payment 1 l. more than at the second, and at the fourth Payment 1 l. more than at the third. The Question is, How much is each Payment?

If you put x for the first Payment, you will have $x + 1$ for the second Payment, $x + 2$ for the third Payment, and $x + 3$ for the fourth Payment: And as all this Money, namely $4x + 6$ ought to be equivalent to 250, you will

will have this Equation, $4x+6=250$, from which substracting 6, you will have $4x=244$, and dividing by 4, you will have $x=61$. Thus you will have 61 *l.* for the first Payment, wherefore the second Payment will be 62 *l.* the third will be 63 *l.* and the fourth will be 64 *l.*

QUESTION IV.

Some Persons having agreed to give 6 Pence a-piece to a Waterman, to carry them from London *to* Gravesend, *on this condition, that if another shou'd come into their Company, he shou'd pay the same Price, and they shou'd share the overplus among them, so that the Waterman shou'd have half, the other half being to be equally divided among the same Persons, or else given to the Waterman, and his Pay to be lessen'd in proportion to what they had promis'd him; There arriv'd a fourth part of their Number, and three over, then the first Comers were to pay but 5 Pence to the Waterman. The number of the Persons that came first is demanded.*

LET $4x$ be the number of the Persons that came first. Then $24x$ is the Money due to the Waterman.

$1x+3$ the Persons that afterwards came.

$6x+18$ the Overplus.

$3x+9$ the half of the Overplus, which must be substracted from $24x$, and there will remain $21x-9$, for the Money due to the Waterman from the first Persons. If then you divide this Money by $4x$, which is the number of the first Persons, you will have $\frac{21x-9}{4x}$ for the Money which each ow'd the Waterman; and as it is suppos'd that each ow'd him 5 Pence, you will have this Equation, $\frac{21x-9}{4x}=5$, which being multiplied by $4x$, you will have this, $21x-9=20x$, and by *Antithesis* you will find $x=9$, and consequently $4x=36$, for the number of Persons sought.

QUESTION V.

Three Ells of Sattin and four Ells of Taffety cost 57 Shillings, and at the same Price 5 Ells of the same Sattin and two Ells of the same Taffety cost 81 Shillings. I demand the value of the Sattin and Taffety per Ell.

If x be put for the value of an Ell of Sattin, and y for the value of an Ell of Taffety, according to the conditions of the Question, you will have these two Equations,

$$3x + 4y = 57$$
$$5x + 2y = 81$$

To the end that in each of these two Equations one of the two unknown Quantities x, y, for example x, may be found multiply'd by one and the same number, which is necessary to be done, that by substracting one Equation from the other, there shou'd remain a third Equation, wherein you have only the other unknown Quantity y; Multiply the first Equation, $3x + 4y = 57$, by the number 5, which multiplies x in the second; and reciprocally the second, $5x + 2y = 81$, by the number 3, which multiplies the same x in the first; and you will have these two other Equations,

$$15x + 20y = 285$$
$$15x + 6y = 243$$
$$\overline{\qquad 14y = 42 \qquad}$$

If you substract the second from the first, you will have this third Equation, $14y = 42$, which being divided by 14, you will have $y = 3$, for the value of an Ell of Taffety. And if in the room of y you substitute its value 3, now found, the first Equation $3x + 4y = 57$, will be chang'd into this, $3x + 12 = 57$, from which substracting 12, and dividing the Remainder $3x = 45$ by 3, you will have $x = 15$, for the Value of an Ell of Sattin.

QUESTION VI.

One Person said to another, if you will give me three of your Crowns, I shall have as much as you have left; and the other answer'd, if you will give me five of yours, I shall have twice as much as you have left: The Question is how many Crowns each Person had.

IF the Letter x be put for the number of Crowns the first Person had, and y for the number of Crowns the second Person had, you will have, according to the conditions of the Question, these two Equations,

$$x + 3 = y - 3.$$
$$y + 5 = 2x - 10$$

In the first, $x + 3 = y - 3$, you will find $y = x + 6$; and in the second, $y + 5 = 2x - 10$, you will find the same $y = 2x - 15$; wherefore you will have this third Equation, $x + 6 = 2x - 15$, in which you'll find $x = 21$, for the Money that the first Person had; and instead of $y = x + 6$, or of $y = 2x - 15$, you will have $y = 27$, for the Money that the other had.

QUESTION VII.

One hundred Persons, consisting of Men, Women and Children, expended in a Feast 100 Pounds or 2000 Shillings; each Man expended 100 Shillings, each Woman 20 Shillings, and each Child 5 Shillings. The Number of Men, Women, and Children is demanded.

IF x be put for the number of the Men, y for the number of the Women, and z for the number of Children, you will have, according the conditions of the Question, these two Equations to be resolv'd,

$$x + y + z = 100$$
$$100x + 20y + 5z = 2000.$$

If from each side of the first, $x + y + z = 100$, you substract x and z, you will have $y = 100 - x - z$, and

if

if in the room of y, you put its value found $100 - x - z$; instead of $20y$ you will have $2000 - 20x - 20z$; and instead of the second Equation $100x + 20y + 5z = 2000$, you will have this $80x - 15z + 2000 = 2000$, from whence substracting 2000, you will have this, $80x - 15z = 0$, and adding $15z$, you will have this, $80x = 15z$, and dividing by 5, you will have this, $16x = 3z$; and lastly dividing by 3, you will have this last Equation, $\frac{16}{3}x = z$, where you see that the Quantity z would be known, if the other Quantity x were also known; and as there is nothing which determines this Quantity x, it shews that the Question propos'd is *Indeterminate*, that is to say, it is capable of an infinite number of different Solutions, because there is liberty to suppose the indeterminate Quantity x whatever one pleases. But there is a Precaution to be taken concerning the value that may be given it, so that the quantity z, or its value found $\frac{16}{3}x$, be a Whole number, which ought to be so in this Question, because the value $\frac{16}{3}x$ represents the number of Children, which ought not to be a Fraction by the nature of the Question. You must suppose then for x a number divisible by 3, which is the Denominator of the Fraction $\frac{16}{3}x$. If therefore you suppose $x = 3$, instead of $\frac{16}{3}x$ for z, you will have 16; and instead of $100 - x - z$ for y, you will have 81. So that 3 Men, 81 Women, and 16 Children, will solve the Question.

To have another Solution, suppose $x = 6$, and then you will find $z = 32$, and consequently $y = 62$; so that 6 Men, 62 Women, and 32 Children, will be a second Solution.

To have a third Solution, suppose $x = 9$, and then you will find $z = 48$, and consequently $y = 43$. So that 9 Men, 43 Women, and 48 Children, will be the third Solution.

To have a fourth Solution, suppose $x = 12$, and then you will find $z = 64$, and consequently $y = 24$. So that 12 Men, 24 Women, and 64 Children, will be the fourth Solution.

To have a fifth Solution, suppose $x = 15$, and then you will find $z = 80$, and consequently $y = 5$. So that 15 Men, 5 Women, and 80 Children, will be the fifth Solution.

There is no other Solution in whole numbers, because by putting for x, a number multiplied by 3, greater than 15, the number of Men, Women, and Children would surpass 100, which is contrary to the Supposition.

QUE-

QUESTION VIII.

A Hall made in the form of Rectangular Parallelogram contains 90 Square Fathoms in its Area, and its Length is twice its Breadth, and three Fathoms more. The Length and the Breadth is demanded.

If x be put for the breadth, you will have by supposition $2x+3$ for the length, which being multiplied by the breadth x, you will have $2x^2+3x$, for the Area of the Rectangle; and as this Area is suppos'd to be 90 Square Fathoms, you will have this Equation, $2x^2+3x=90$, which being divided by 2, you will have this, $x^2+\frac{3}{2}x=45$. Add to each side the Square $\frac{9}{16}$ of the half $\frac{3}{4}$ of the Coefficient $\frac{3}{2}$ of the second Term, and you will have this Equation, $x^2+\frac{3}{2}x+\frac{9}{16}=\frac{729}{16}$, whose square Root will give this Equation in lower Terms, $x+\frac{3}{4}=\frac{27}{4}$, from which substracting $\frac{3}{4}$ you will have $x=6$, for the breadth sought; and instead of $2x+3$, you will have 15 for the length. Thus the length of the Rectangle which was sought for, will be 15 Fathoms, and its breadth will be 6.

THE

THE PRACTICE OF Geometry.

OUR Design is to add here only the most useful and most easy Problems for Practice, whether on the Ground, or on Paper only; for the use of Beginners, to dispose them the better to understand what we have to say hereafter, which requires a further knowledge, without taking the pains of adding here the Definitions of many common Terms, which are generally well enough understood by every body, or which may be understood without any difficulty by the Practices hereafter taught, till such time as these Terms be explain'd and defin'd in their place.

PROBLEM I.

To draw a Right Line from one given Point to another, upon a Plane.

Plate I. Fig. I. FIrst, if the two Points be given upon Paper, or upon some other Plane of a small extent, as A, B, it is naturally known by every one, that there is nothing to do but to apply a Ruler upon the two given Points A, B, and draw a Right Line with a Pin or Pencil along the Ruler.

Secondly,

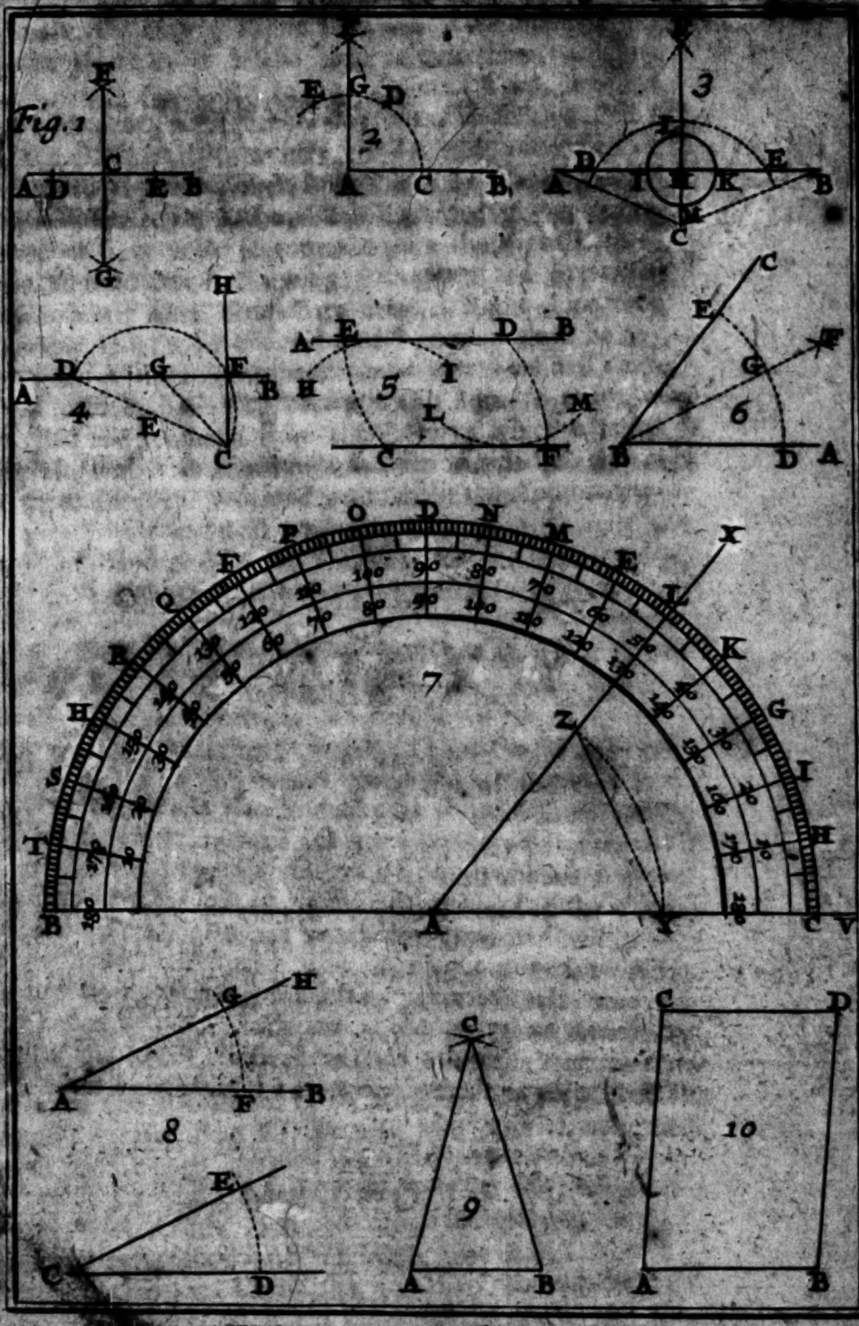

Fig. 1
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Secondly, to draw a Right Line thro' two Points given upon the Ground, it is also evident that there needs no more than to apply to the two given Points a Cord, stretch'd out at both ends, as Artificers do, when these two Points are not far distant; otherwise 'tis done by a visual Ray, guided by the sights of some Instrument, by planting Stakes at proper distances along the visual Ray, and giving notice, by word or sign, when it removes from the Right Line. Plate 1. Fig. 1.

This Method is usual among Surveyors and Engineers, that frequently have occasion to draw a Right Line of a considerable length on the Ground: And if there be any danger, as when an Engineer would carry on a Trench towards a Place besieged, he traces this Line by means of a Fire, hid and conceal'd from the Enemy, which is set at a place pitch upon in the day-time, and which he aims to come at, to direct the Workmen, and make the Approaches.

PROBLEM II.

To draw a Perpendicular to a given Line, thro' a given Point.

THree Cases may happen; for the given Point may be either in the given Line, or at one of the two extremities of the given Line, or out of the given Line. And moreover, the Point and the Line may be given either upon Ground or upon Paper. We shall first work upon Paper with Rule and Compass, and proceed in the same manner on the Ground with Cord and Stake.

First then, if the Point C be given in the given Line AB, to draw a Perpendicular thro' this given Point C, take at pleasure from the given Point C, upon the given Line AB on both sides, the two equal Distances CD, CE, and describe from the two Points E, D, with any opening of the Compasses greater than CD or CE, two Arcs of a Circle on both sides, which intersect here at the two Points F, G, thro' which you must draw the Right Line FG, which if the work is done right, will pass thro' the given Point C, and will be perpendicular to the given Line AB.

When you have no Compasses, you may make use of a Square, by applying its Right Angle to the given Point C, so that one of its sides may precisely answer one of the two Parts AC, BC, as for example upon the part AC, and then you must draw along the other side thro' the given Point C, the Perpendicular CF, which is sought for: And to know if it is well drawn, likewise to know if the Square Fig. 3.

be

Plate 1. Fig. 1. be good, you must apply one of its sides on the other part BC, for then the other side ought to coinside with the Perpendicular CF.

When the Line AB is given on the Ground, you must describe from the two Points E, D, two Arcs of a Circle, with Cords of any length, but equal, and greater than one of the two Lines CD, CE; and as it is sometimes inconvenient to describe Arcs of a Circle upon the Ground, it will be better to join the two ends of these Cords together, which ought to be equally stretch'd out, to have the point F, thro' which, and thro' the given point C, you may draw the Perpendicular CF.

You may also draw this Perpendicular CF, by making at the given Point C, with a Graphometer, (*Theodolite*) or otherwise, an Angle of 90 degrees, as will be taught in *Prob.* 9. You may do the same thing upon Paper with a Protractor, or with a Sector, or otherwise, as will be also taught in *Prob.* 9.

Fig. 2. Secondly, if the Point thro' which you are to draw a Perpendicular to the Line AB, is given in one of its extremities, as A, describe at pleasure from this Point A, the Arc of a Circle CDE, and with the same opening of the Compass, set off twice from the Point C, where it cuts the Line AB in D, and from D, in E, describe from the two Points E, D, still with the same opening of the Compass, two Arcs of a Circle which cut here in the Point F, thro' which, and thro' the given Point A, draw the Right Line AF, which will be Perpendicular to the propos'd Line AB.

This Perpendicular may also be drawn by the means of a Square, or by making at the given Point A, an Angle of 90 degrees. But we shall teach another Method to do the same in *Prop.* 31. *l.* 3. of *Euclid's Elements.*

When you are to draw a Perpendicular upon the Ground, you may also make at the end A of the Line AB, an Angle of 90 Degrees; or you may do as will be taught in *Prop.* 48. *l.* 1. and likewise in *Prop.* 31. *l.* 3. of *Euclid's Elements.*

Fig. 3. Lastly, if the Point thro' which you are to draw the Perpendicular, be given out of the given Line AB, as C, describe at pleasure from this Point C, the Arc of a Circle DE, which cuts the given Line AB in two points, as DE, from which describe with the same opening of the Compass,

pass, two Arcs of a Circle, and draw thro' their Intersection F, and the given Point C, the Right Line CF, which will be the Perpendicular required. Plate 1.

It may happen that the given Point C shall be so nigh one of the two ends of the given Line AB, that it will be hard to describe a Circle which will conveniently cut it in two Points; in this case draw through the given Point C, towards the other end, the Right Line CD, which you are to divide into two equal parts in the Point E; from E describe thro' the two Points C, D, the Semicircle CFD, which will cut the given Line AB in the Point F, thro' which the Perpendicular CF ought to pass. Fig. 4.

When the given Point C is upon the Ground, describe, with a Cord, an Arc of a Circle, so as to cut the given Line AB in two equal parts, as D, E, and divide the Line DE in two equal parts in the Point H, thro' which, and thro' the given Point, draw the Perpendicular CH. Fig. 3.

If the Cord cannot conveniently cut the given Line AB in two Points, which will happen when the given Point C shall be towards one of the two ends of the Line AB, you must extend it towards the other end, until it meets the Line AB in some point, as D, and having divided it in two equal parts at the Point E, you must extend its half EC, or ED, from E, until it meets the given Line AB in one Point, as F, thro' which you may draw the Perpendicular CF. Fig. 4. Fig. 4.

Or describe thro' the given Point C, from the two Points G, D, taken at pleasure upon the given Line AB, with a Cord, if you work on the Ground, or with a Compass is you work upon Paper, two Arcs of a Circle, which cut each other at the Point H, thro' which, and thro' the given Point C, draw the Perpendicular CH.

If you cannot conveniently trace Arcs of a Circle upon the Ground, tye at the given Point C a Cord, and extend it until it touches the given Line AB, then measure the length of it exactly, which will give the Quantity of the Perpendicular CF, which we will suppose 6 Fathoms; Then seek a square number, from which substracting the square of 6, that is to say, 36, the remainder is a square number. This first and greatest square number is 100, whose side 10 will represent the length of the Line CD; for if from 100 you substract 36, there remains 64, whose square Root is 8, which represents the length of the part DF, the Perpendicular CF

Plate 1. Fig. 4. being 6, as we have already said. Tye then at the given Point C, a Cord 10 Fathoms long, and extend it till its extremity meets the given Line AB in some point, as in D, from whence you must reckon upon the given Line AB, towards the given point C, 8 Fathoms, for example as far as the Point F, thro' which you may draw the Perpendicular CF.

To find a square number, from which substracting a given square number, there remains a square number, use this general Canon, which we have drawn from *Algebra*.

If to the given Square an indeterminate Square be added, greater or less than the given Square, and if the Sum be divided by double the Side of the same indeterminate Square, you will have the Side of the Square sought.

As if to the given square 36, the square 4 be added, whose side is 2, and if by the double 4 of this side 2, you divide the sum 40, the quotient 10 will give the side of the square sought, or the length of the Line DF.

In like manner, if to the same given square 36, the square 9 be added, whose side is 3, and the sum 45 be divided by the double 6 of the same side 3, you will have 7 fathoms and 3 feet for the line DF, and then the line CD will be 4 fathoms and 6 feet.

All these practices are only proper upon the Ground, when the given point C is not very remote from the given line AB; for when the distance of this point is great, Cords cannot be conveniently used, which even tho' they may be long enough, yet cannot be easily extended. In this case, a *Theodolite* or some other Surveying-Instrument may be used thus.

Fig. 3. To draw then from the given Point C upon the Ground, a Perpendicular to the given Line AB, fix the Staff upon this Line AB, and turn the Instrument about, looking along the Diameter IK, till you see the two ends A, B, of the same Line AB, and then this Diameter IK will precisely answer upon the Line AB; and holding the Instrument in this situation, you must change it from the place by advancing it to the right or to the left, until by the other perpendicular Diameter LM, you may see the given point C; and the point H where the Staff remains, will be

be that thro' which, and thro' the given Point C, you may draw the Perpendicular CH. Plate 1.

The Surveying Instrument may be let alone, by imagining from the given Point C, to the two Points, as A, B, taken at pleasure upon the given Line AB, the two Lines CA, CB, drawn; so that the given Point C be, if it is possible, between the two Points A, B, that is to say, that the Perpendicular CH, be between the two Lines CA, CB, or within the Triangle ABC, whose three sides ought to be measur'd exactly, and by their means to find the distance from the point H, of the Perpendicular, to one of the two points A, B, as A, answering to the side AC, which I suppose the greater; and it may be done thus: Fig. 3.

Divide by the double of the Base AB of the Triangle ABC, the excess of the sum of the square of the same Base AB, and of the square of the greater side AC, above the square of the less BC.

Thus if the greater side AC be of 15 fathoms, the less BC 13, and the base AB 14, by dividing the excess 252, of the sum 421, of the squares AB, AC, above the square BC, by the double 28 of the base AB, you will have 9 fathoms for the distance from the point H of the Perpendicular, to the point A. If then you reckon 9 fathoms from A to H, and you draw the right line CH, it will be the Perpendicular sought.

If you cannot conveniently chuse upon the given Line AB, two points, between which is the point F of the Perpendicular, as if you could only take the two points A, G, so that the Perpendicular CF falls without the Triangle ACG, whereof the sides AG, AC, CG, ought likewise to be known; you may find the distance FG, from the point F of the Perpendicular, to the nearest point G, thus: Fig. 4.

Divide by double the base AG, the excess of the square of the greatest side AC, above the sum of the squares of the two other sides AG, CG.

Thus if the greater side AC were 15 fathoms, the Base AG 4, and the other side CG 13, by dividing the excess 40 of the square AC, which is 225, above the sum 185, of the squares 16, 169, of the two other sides AG, CG, by the double 8 of the base AG, you will have 5 fathoms for

the diſtance FG, &c. We will give in *Prop.* 15. *l.* 1. of *Euclid's Elements*, another method of drawing a Perpendicular.

PROBLEM III.

Thro' a given Point to draw a Right Line, parallel to a given Right Line.

Plate 1. Fig. 5. THRO' the given Point C to draw a Line parallel to the given Line AB; from the Point D taken at pleaſure in the Line AB, thro' the point C deſcribe the Arc CE, and from the Point C thro' the Point D, the Arc DF, equal to the preceding CE, and you have the Point F, thro' which, and the given Point C, draw the Right Line CF, which will be parallel to the given Line AB.

Or from the given Point C deſcribe the Arc HI, touching the given Line AB, and from the Point D, taken at pleaſure in the ſame Line AB, deſcribe with the ſame opening of the Compaſs, the Arc LM: Laſtly, thro' the given Point C, draw the Right Line CF, touching the Arc LM, which will be the parallel requir'd. When it is to be perform'd on the Ground, do as is taught in *Prop.* 31. *l.* 1. of *Euclid's Elements*. We ſhew in *Prop.* 34. *l.* 1. of the ſame *Elements*, another method, how upon Paper to draw a Parallel to a given Line thro' a given Point: and in *Prop.* 21. *Book* 3. of the ſame *Elements*, we ſhew how to draw thro' a given Point, a Line parallel to a given inacceſſible Line upon the Ground.

PROBLEM IV.

To divide a given Right Line into two equal parts.

Fig. 1. TO divide the given Line AB into two equal parts; deſcribe from its two ends A, B, with one and the ſame opening of the Compaſs, two Arcs interſecting at the two Points F, G, thro' which draw the Right Line FG, which will divide the given Line into two equal parts in the point C.

'Tis in the ſame manner that you muſt work it on the Ground, by deſcribing the Arcs with two Cords of the ſame length, tied to the two ends A, B; but to ſave the trouble

trouble of describing Arcs, (which is pretty hard when the Ground is very uneven, and full of Thorns or Briars) join the two ends of those two Cords, on one side and the other, and you will have the two points F, G; or more easily extend a Cord along the Line AB, and redouble it by joining its two ends, for thus you will have the half of the the given line AB, and then there needs no more than to set off this half or redoubled Cord along the line AB, from one of its ends A or B, to find C the middle point requir'd. Plate 1. Fig. 1.

If the Cord be less than the given line AB, cut off the two equal parts AD, BE, and divide the line DE into two equal parts.

PROBLEM V.

To divide a given Arc of a Circle into two equal parts.

TO divide the arc DE of a Circle whose Center is B, into two equal parts, describe from its two ends E, D, with one and the same opening of a Compass, two arcs intersecting each other in the point F, from which to the Centre B, draw the right line BF, which will divide the given arc DE into two equal parts at the point G. Fig. 6.

When we say that two arcs of a Circle must be describ'd with one and the same opening of a Compass, without particularizing any thing, it is to be understood that this opening may be taken at pleasure, provided the two arcs intersect. Fig. 5.

If the Centre of the given arc DE were not likewise given, you might divide it into two equal parts, by means of the preceding Problem, as if this arc were a right line.

PROBLEM VI.

To divide a given Angle into two equal parts.

TO divide the given angle ABC into two equal angles; describe from the angular point B, the arc DE, with any opening of the Compass, the greater the better, and from the two ends E, D, with one and the same opening of the Compass, describe two arcs intersecting in the point F, thro' which, and the point B, draw the right line BF, which will divide the given angle ABC into two equal parts, that is to say, the two angles ABF, CBF, will be equal to each other, as well as the two arcs GD, GE, which measure 'em. Fig. 6.

When the angle ABC is given upon the Ground, one may find how many degrees it is of, as is ſhewn in *Prob.* 8. and by *Prob.* 9. make at the angular point B, with the line AB, or with the line BC, an angle equal to the half of the propoſed angle ABC, by means of the right line BF, which conſequently will divide the angle ABC into two equal parts.

PROBLEM VII.

To divide the Circumference of a Circle into Degrees.

MAthematicians divide the Circumference of a Circle into 360 equal parts, which they call *Degrees*; each Degree into 60 equal parts call'd *Minutes*; each Minute into 60 other equal parts, which they call *Seconds*; and ſo on. They have choſen the number 360 for the Circle, and the number 60 for the ſubdiviſions, becauſe theſe two numbers have ſeveral aliquot parts, and ſo are more convenient in the Practice. We ſhall content our ſelves with the diviſion of the Semicircle into 180 degrees, as being ſufficient for what we have need of.

Plate 1. Fig. 7. Having from the point A, taken at pleaſure in the indefinite line BC, deſcribed the Semicircle BDC, firſt divide its Circumference into three equal parts, by ſetting off the ſame opening of the Compaſs, that is to ſay, the length of the Semidiameter AB or AC, from C to E, and from E to F, or from B to F, and from F to E, and you'll have the three equal parts CE, EF, FB, whereof each is equivalent to 60 degrees. Divide the arc CE into two equal parts in the point G, the arc EF into two equal parts in the point D, the arc FB into two equal parts in the point H, and the Semicircle will be divided into ſix equal parts, each of which will be equivalent to 30 degrees. Divide the arc CG into three equal parts in the points H, I, the arc GE into three equal parts in the points K, L, the arc ED into three equal parts in the points M, N, the arc DF into three equal parts in the points O, P, the arc FH into three equal parts in the points Q, R, and the arc BH into three equal parts in the points S, T; and the Semicircle will be divided into eighteen equal parts, each of which comprehends 10 degrees; wherefore if you divide each of theſe eighteen equal parts into two other equal parts, the Semicircle will be divided into thirty ſix equal parts,

parts, each of which being lastly divided into five equal parts, the Semicircle will be divided into its 180 degrees, to which you must annex figures from 10 to 10 degrees, as you see in the Scheme which represents that Semicircle which Instrument-makers do commonly make upon Brass, and which they call a *Protractor*, or *Transporter*, because by applying it upon an angle, the quantity of that angle may be measur'd, or by applying it upon a given line, an angle of as many degrees as you will may be made, as we shall shew in the following Problems.

PROBLEM VIII.

To find how many Degrees a given Angle contains.

AS the measure of a rectilineal angle is the arc of any Circle describ'd from its angular point, it follows, that if the number of the degrees compris'd between the lines which form the angle be known, the value of this angle will be known also. Wherefore if it is propos'd to measure the angle VAX, apply the Protractor upon this angle, so that its Centre may lye upon the angular point A, and its Diameter AC upon one of the two lines which form the angle, as upon the line AV, and then the arc CL of the Protractor, compris'd between the two lines forming the angle, being here of 50 degrees, shews that the given angle VAX is 50 degrees. Plate 1. Fig. 7.

If you have no *Protractor*, make use of the *Sector*, thus; Having describ'd at pleasure from the angular point A of the given angle VAX, the arc YZ, set off the same opening AY or AZ upon the Line of Chords of the Sector, from 60 to 60; and the Sector remaining thus open, set off upon the same Line of Chords the arc YZ, and the equal number of degrees on both sides that this extends, will give the quantity of the arc YZ, and consequently of the given angle VAX.

If the angle be given on the Ground, whether really or imaginarily, measure it by means of a large *Semicircle* divided exactly into 180 degrees, and sometimes into Minutes, or at least into every 5 Minutes. This *Semicircle*, which the *Swedes* and *Germans* commonly call *Astrolabe*, and the *French* call *Graphometer*, is commonly made of Brass, and has an *Alidade* or Index, being a Ruler of the same

 Metal,

Plate I. Fig. 7. Metal, made to move about the Centre of the Semicircle, with two sights set up at right angles, so that the holes, or fine slits, which serve to direct the visual Rays, correspond to the *Line of Direction*, which is drawn upon the Alidade or Index, and passes thro' the Centre of the Instrument, where the visual angles are form'd.

This Instrument has also two sights set up at right angles, each near one of the two ends B, C, of the Diameter BC, and the slits of these sights serve also to conduct the Eye along the Diameter BC. This Instrument is so common, that it doesn't seem necessary to give a longer description of it, wherefore I shall teach at present how to use it, to measure an accessible angle upon the Ground.

To measure then upon the Ground the accessible angle VAX, apply on this angle the Semicircle, which ought to be sustain'd by a Staff, so that its Centre answers perpendicularly upon the angular point, which may be easily done with a Plummet; and holding the Instrument almost parallel to the Plane of the given angle, turn it about till you see thro' the immoveable sights some point of the line AV, for thus the Diameter BC will answer upon this line AV, which ought to be so always; and the Instrument being fixt in this situation, turn the Index, until thro' the sights thereof you see some point of the other line AX, and then the Line of Direction will shew upon the Circumference of the Semicircle the number of degrees in the given angle VAX.

An accessibe angle on the Ground may be also very easily and very exactly measur'd by means of the following Table, which shews the degrees and minutes of the angles, whose two sides are each 30 feet, and the Bases being right lines, encrease by two and two Inches only, and this is sufficient for practice.

Table

Table of Plane Angles comprehended by two Sides of 30 Feet.

Bases.		Angles.		Bases.		Angles.		Bases.		Angles.	
Fe.	Inc.	D.	M.	Fe.	Inc.	D.	M.	Fe.	Inc.	D.	M.
0	0	0	0	5	0	9	34	10	0	19	11
0	2	0	19	5	2	9	53	10	2	19	30
0	4	0	38	5	4	10	12	10	4	19	50
0	6	0	57	5	6	10	31	10	6	20	19
0	8	1	8	5	8	10	50	10	8	20	29
0	10	1	36	5	10	11	9	10	10	20	48
1	0	1	55	6	0	11	29	11	0	21	8
1	2	2	14	6	2	11	48	11	2	21	27
1	4	2	33	6	4	12	8	11	4	21	46
1	6	2	52	6	6	12	27	11	6	22	6
1	8	3	11	6	8	12	46	11	8	22	25
1	10	3	30	6	10	13	5	11	10	22	45
2	0	3	49	7	0	13	24	12	0	23	5
2	2	4	8	7	2	13	43	12	2	23	24
2	4	4	28	7	4	14	2	12	4	23	44
2	6	4	47	7	6	14	22	12	6	24	3
2	8	5	6	7	8	14	41	12	8	24	32
2	10	5	25	7	10	15	0	12	10	24	52
3	0	5	44	8	0	15	20	13	0	25	1
3	2	6	3	8	2	15	39	13	2	25	21
3	4	6	22	8	4	15	58	13	4	25	41
3	6	6	41	8	6	16	18	13	6	26	1
3	8	7	0	8	8	16	37	13	8	26	20
3	10	7	20	8	10	16	56	13	10	26	40
4	0	7	39	9	0	17	15	14	0	26	59
4	2	7	58	9	2	17	34	14	2	27	18
4	4	8	17	9	4	17	54	14	4	27	38
4	6	8	36	9	6	18	13	14	6	27	58
4	8	8	55	9	8	18	32	14	8	28	18
4	10	9	14	9	10	18	52	14	10	28	38

Table of Plane Angles comprehended by two Sides of 30 Feet.

Bases.		Angles.		Bases.		Angles.		Bases.		Angles.	
Fe.	Inc.	D.	M.	Fe.	Inc	D.	M.	Fe.	Inc.	D.	M.
15	0	28	57	20	0	38	56	25	0	49	15
15	2	29	17	20	2	39	17	25	2	49	36
15	4	29	37	20	4	39	38	25	4	49	57
15	6	29	56	20	6	39	58	25	6	50	18
15	8	30	16	20	8	40	18	25	8	50	39
15	10	30	35	20	10	40	38	25	10	51	0
16	0	30	56	21	0	40	59	26	0	51	21
16	2	31	16	21	2	41	19	26	2	51	42
16	4	31	36	21	4	41	40	26	4	52	3
16	6	31	56	21	6	42	0	26	6	52	24
16	8	32	16	21	8	42	20	26	8	52	46
16	10	32	35	21	10	42	40	26	10	53	8
17	0	32	55	22	0	43	1	27	0	53	29
17	2	33	15	22	2	43	22	27	2	53	51
17	4	33	35	22	4	43	42	27	4	54	12
17	6	33	55	22	6	44	3	27	6	54	34
17	8	34	15	22	8	44	24	27	8	54	55
17	10	34	35	22	10	44	44	27	10	55	16
18	0	34	55	23	0	45	5	28	0	55	38
18	2	35	15	23	2	45	26	28	2	56	0
18	4	35	35	23	4	45	46	28	4	56	22
18	6	35	55	23	6	46	7	28	6	56	43
18	8	36	15	23	8	46	28	28	8	57	5
18	10	36	35	23	10	46	48	28	10	57	26
19	0	36	55	24	0	47	9	29	0	57	48
19	2	37	15	24	2	47	30	29	2	58	10
19	4	37	36	24	4	47	51	29	4	58	32
19	6	37	56	24	6	48	12	29	6	58	54
19	8	38	16	24	8	48	33	29	8	59	16
19	10	38	36	24	10	48	54	29	10	59	38

Table of Plane Angles comprehended by two Sides of 30 Feet.

Bases.		Angles.		Bases.		Angles.		Bases.		Angles.	
Fe.	*Inc.*	*D.*	*M.*	*Fe.*	*Inc.*	*D.*	*M.*	*Fe.*	*Inc.*	*D.*	*M.*
30	0	60	0	35	0	71	22	40	0	83	37
30	2	60	22	35	2	71	46	40	2	84	3
30	4	60	44	35	4	72	10	40	4	84	29
30	6	61	6	35	6	72	33	40	6	84	54
30	8	61	28	35	8	72	56	40	8	85	20
30	10	61	50	35	10	73	20	40	10	85	46
31	0	62	13	36	0	73	44	41	0	86	13
31	2	62	35	36	2	74	8	41	2	86	39
31	4	62	58	36	4	74	32	41	4	87	5
31	6	63	20	36	6	74	56	41	6	87	32
31	8	63	43	36	8	75	20	41	8	88	58
31	10	64	5	36	10	75	44	41	10	88	25
32	0	64	28	37	0	76	9	42	0	88	51
32	2	64	50	37	2	76	33	42	2	89	18
32	4	65	13	37	4	76	57	42	4	89	44
32	6	65	36	37	6	77	22	42	6	90	12
32	8	65	58	37	8	77	46	42	8	90	39
32	10	66	21	37	10	78	9	42	10	91	6
33	0	66	44	38	0	78	34	43	0	91	33
33	2	67	7	38	2	79	0	43	2	92	1
33	4	67	30	38	4	79	25	43	4	92	29
33	6	67	53	38	6	79	50	43	6	92	56
33	8	68	16	38	8	80	14	43	8	93	24
33	10	68	39	38	10	80	40	43	10	93	52
34	0	69	2	39	0	81	5	44	0	94	20
34	4	69	25	39	2	81	30	44	2	94	48
34	2	69	48	39	4	81	55	44	4	95	16
34	6	70	12	39	6	82	20	44	6	95	45
34	8	70	35	39	8	82	46	44	8	96	13
34	10	70	59	39	10	83	12	44	10	96	42

Table of Plane Angles comprehended by two Sides of 30 Feet.

Bases.		Angles.		Bases.		Angles.		Bases.		Angles.	
Fe.	Inc.	Deg.	M.	Fe.	Inc.	Deg.	M.	Fe.	Inc.	Deg.	M.
45	0	97	11	50	0	112	53	55	0	132	3
45	2	97	40	50	2	113	28	55	2	133	44
45	4	98	9	50	4	114	4	55	4	134	30
45	6	98	38	50	6	114	38	55	6	135	20
45	8	99	8	50	8	115	14	55	8	136	11
45	10	99	37	50	10	115	49	55	10	137	3
46	0	100	6	51	0	116	26	56	0	137	7
46	2	100	36	51	2	117	2	56	2	138	49
46	4	101	6	51	4	117	39	56	4	139	44
46	6	101	36	51	6	118	16	56	6	140	30
46	8	102	7	51	8	118	53	56	8	141	38
46	10	102	37	51	10	119	31	56	10	142	6
47	0	103	8	52	0	120	9	57	0	143	36
47	2	103	39	52	2	120	47	57	2	144	39
47	4	104	10	52	4	121	26	57	4	145	43
47	6	104	41	52	6	122	6	57	6	146	48
47	8	105	12	52	8	122	45	57	8	147	57
47	10	105	44	52	10	123	25	57	10	149	8
48	0	106	16	53	0	124	6	58	0	150	20
48	2	106	48	53	2	124	47	58	2	151	36
48	4	107	20	53	4	125	28	58	4	152	55
48	6	107	52	53	6	126	10	58	6	154	19
48	8	108	25	53	8	126	52	58	8	155	48
48	10	108	57	53	10	127	35	58	10	157	22
49	0	109	30	54	0	128	19	59	0	159	3
49	2	110	4	54	2	129	3	59	2	160	53
49	4	110	37	54	4	129	48	59	4	162	54
49	6	111	11	54	6	130	33	59	6	165	12
49	8	111	44	54	8	131	19	59	8	167	48
49	10	111	18	54	10	132	6	59	10	171	28

If then it is propos'd to find the quantity of the angle VAX, take on each of its two sides AV, AX, the two parts AY, AZ, each of 30 feet, and measure the base YZ exactly in feet and inches, which we will suppose of 25 feet 6 inches, to which there answers in the Table 50 degrees 18 Minutes, for the quantity of the propos'd angle VAX. Plate 1, Fig. 7.

The same Table may also be of use to measure the same angle VAX, when it is upon Paper, namely by taking on the two sides AV, AX, of the angle, the two parts AY, AZ, each of 30 equal parts from some *Scale*, that is to say, upon a line divided exactly into equal parts, and by setting off the base YZ upon the same scale, you'll know how many like equal parts it contains, for this number of equal parts being sought in the Column of bases in the preceding Table, will give on the other side in another Column, the degrees and minutes that the angle VAX contains.

PROBLEM IX.

At a given Point on a given Line, to make an Angle of a given Magnitude.

AT the given point A, upon the given line AV, to make an angle, for example, of 50 degrees; apply the Diameter of the Protractor on the given line AV, so that its Centre answers exactly on the given point A, and the Instrument remaining so fixt, reckon from the extremity C, of its Diameter, the 50 degrees proposed, and where they terminate, mark the point L, thro' which, and thro' the given point A, draw the right line ALX, which will make with the given line AV, the angle VAX, of 50 degrees. Fig. 7.

If the point A is given upon the Ground, we use the *Graphometer* or *Theodolite*, and place it in such a manner, that it may have a situation almost parallel to the given Line AV, that its Centre answers perpendicularly on the given point A, and that its Diameter BC answers on the line AV, which will happen, when by looking thro' the immoveable sights, you see some point of the given line AV, then the Instrument being so fix'd, and the Index being turn'd to the point L of 50 degrees, since an angle of 50 degrees is to be laid down, plant a Stake in the Ground in a point as X, which is in the visual line passing thro' the sights of the Index, that is to say, so that this Stake

Plate 1. Fig. 7. Stake being stuck upright, may be perceiv'd by looking thro' the sights of the Index, and then the line imagined to pass by the point X, and by the given point A, will make with the given line AV an angle of 50 degrees, as was requir'd.

You may also, by means of the preceding Table, make on the Ground any angle you please, on a given point of a given line; as if at the point A, of the given line AV, you would make with the same line AV, an angle, for example, of 56 degrees, reckon 30 feet on this line AV, from A to Y, and there plant a Stake, to which tie a Cord 28 feet and 2 inches long, such as you find the base of an angle of 56 degrees to be in the preceding Table: plant also at the point A another Stake, to which tye another Cord equal to the line AY, that is to say, 30 feet long; lastly, join the two ends of these two Cords, tied to their Stakes, by extending them so that each side be fully stretch'd out, and plant a Stake where the two ends, being join'd together, meet upon the Ground, as in Z; and then the imaginary line AZ, will make with the propos'd line AV, which is often no other than imaginary, an angle of 56 degrees, as was required.

The same Table will also serve to make upon Paper the same angle of 56 degrees, or of any other number of degrees you please, by describing from the given point A the arc YZ, with the distance of 30 equal parts, taken off from some Scale, and set off on this arc the line YZ of 28 equal parts taken off from the same Scale, and you have the point Z, thro' which, and thro' the given point A, draw the line AZX, which will make with the given line AV, the angle VAX of 56 degrees.

But the *Sector* may serve also very conveniently to make upon Paper an angle of any number of degrees, as for example of 50 degrees, thus; describe from the given point A the arc YZ, with any opening of the Compass, which set off on the two Lines of Chords of the *Sector*, from 60 to 60, so that the *Sector* be so open'd, that the distance from 60 to 60 on the Chords, be equal to the Semi-diameter AY, and the *Sector* remaining thus open, take off the same Chords the distance from 50 to 50, since you would have an angle of 50 degrees, and set it on the arc YZ, from Y to Z, and the arc YZ will be 50 degrees, wherefore by drawing the line AZX, the angle VAX will be 50 degrees.

You

You may also make on the Ground an angle of as many degrees as you will, by the help of the *Sector*, which for this purpose ought to have two sights fitted at right angles to each Line of Chords to direct the visual Rays, with which you may make what angle you will, by opening the *Sector* in such a manner, that the two Lines of Chords shall make the same angle at the Centre of the *Sector*, which ought to answer to the point given on the Ground; and this may be done by setting off from the Centre on one of the two Lines of Chords, the distance of the Chord correspondent to the number of degrees proposed, and applying the length of this Chord upon the same Lines of Chords, on both sides from 60 to 60; for thus the *Sector* will be found open as is requir'd. See our *Treatise of the Use of the Sector, or Compass of Proportion.* Plate 2. Fig. 7.

PROBLEM X.

At a given Point of a given Line to make an Angle equal to an Angle given.

AT the given point A of the given line AB, to make an angle equal to the given angle C; describe from this angle C, with any opening of the Compass, the arc DE, and with the same opening, from the given point A describe the arc FG, equal to the first DE, and you will have the point G, thro' which, and the given point A, draw the right line AGH, which will make the angle BAH, equal to the given angle C. Fig. 8.

When you work on the Ground, you must, by *Prob.* 8. measure how many degrees the propos'd angle C contains, and by *Prob.* 9. make at the given point A, the angle BAH, of as many degrees as is the angle C; for thus these two angles will be equal, and the Problem resolv'd.

PROBLEM XI.

Upon a given Line to make an Isosceles Triangle.

TO describe upon the given line AB an Isosceles Triangle; describe from its two ends A, B, with one and the same opening of the Compass two arcs, and thro' their point Fig. 9.

Plate 1. Fig. 9. point of Intersection C, draw to the same extremities A, B, the right lines AC, BC, and the Triangle ABC will be Isosceles; But this Triangle will be equilateral, when the two arcs are drawn with an opening of the Compass equal to the given line AB.

Work in the same manner when the line AB is given on the Ground, to wit, by tying to the two ends A, B, two Cords of one and the same length, and describe by their means two arcs; or if these two arcs cannot conveniently be describ'd, join the ends of these two Cords equally stretch'd out, and you have the Vertex C of the Triangle sought.

PROBLEM XII.

To make a Parallelogram with two given Lines.

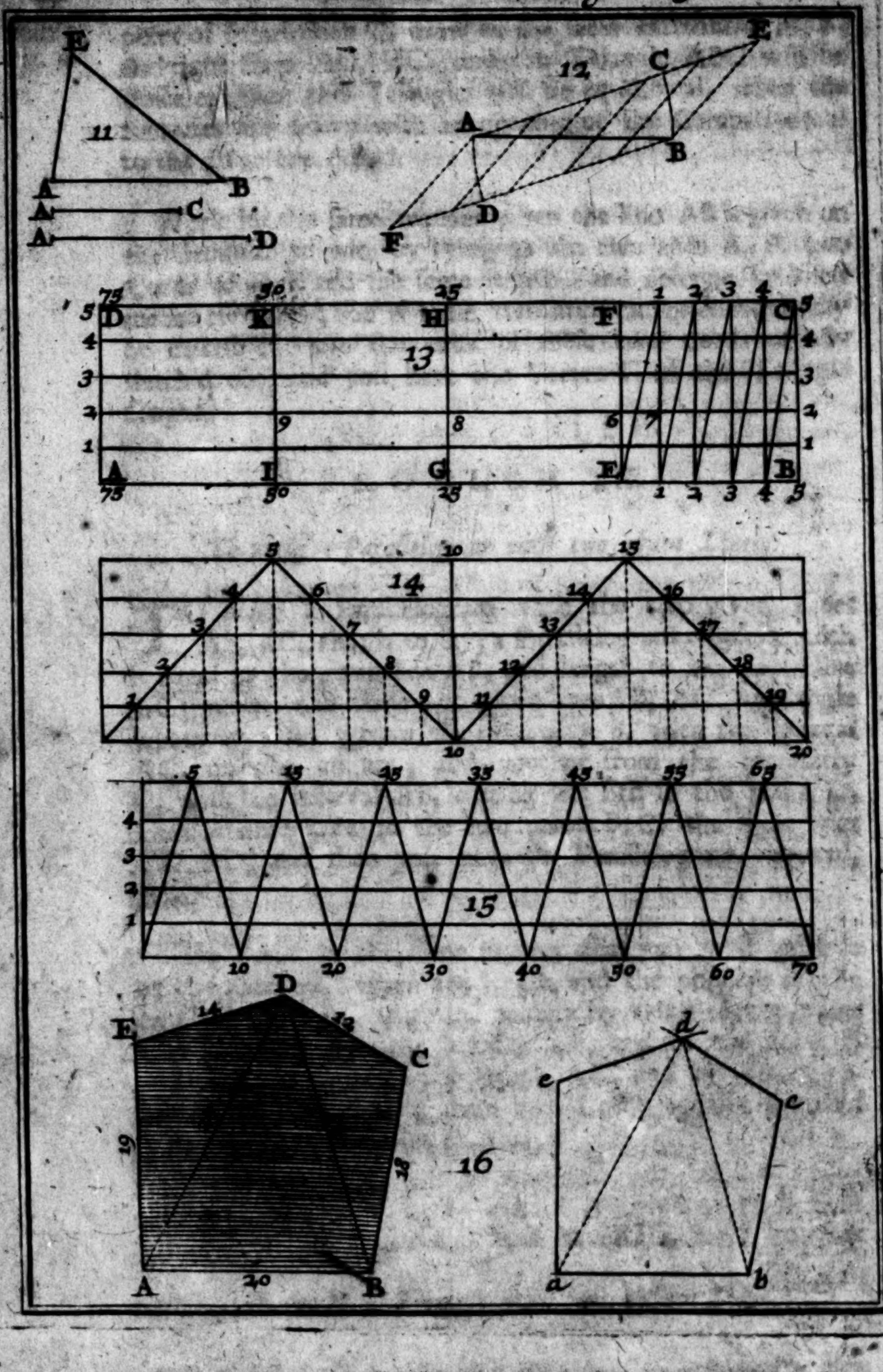
Fig. 10. TO make a Parallelogram with the two given Lines AB, AC, that is to say, a Parallelogram whose breadth is equal to the given line AB, and length to the given line AC; make with these two given lines AB, AC, any angle whatever, BAC; from the extremity B, with the interval AC, describe an arc, and another from the extremity C, with the interval AB, cutting the first in the point D, from whence draw to the two points B, C, the right lines CD, BD, and then you have the Parallelogram required, ABDC.

'Tis almost in the same manner that you must work it on the Ground, when the length and the position of the two lines AB, AC is given, namely by tying to the point C a Cord equal to the breadth AB, and to the point B another Cord equal to the length AC, and by joining together the two ends of these two Cords equally stretch'd out, you have the point D, &c.

PROBLEM XIII.

To make a Triangle with three given Lines.

Plate 2. Fig. 11. TO make a Triangle with the three given lines AB, AC, AD, the greatest of which ought to be less than the sum of the other two; from the extremity A of the first given line AB, with the second given line AC in the Compasses,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passes, describe an arc, and another from the other extremity B, with the third given line AD in the Compasses; and thro' the intersecting point E of these two arcs, draw to the same extremities A, B, the right lines AE, BE, and the Triangle ABE will be that requir'd. Plate 2. Fig. 11.

When you work on the Ground, tie to the extremity A of the first given line AB, a Cord equal to the second AC, and to the other extremity B, another Cord equal to the third AD, then join together the ends of those two Cords equally stretch'd out, and you will have the point E, *&c.*

PROBLEM XIV.

To divide a given Line into any number of equal parts.

TO divide the given line AB, for example, into five equal parts; describe from the extremity A thro' the other extremity B, the arc BC, and from the extremity B, thro' the other extremity A, the arc AD equal to the arc BC, which may be of what bigness you please, and from the two Extremities A, B, thro' the points C, D, draw the indefinite lines ACE, BDF, which will terminate in E and F, by running over on each from the two extremities A, B, five equal parts of any bigness, but the same on the one and the other line; lastly, draw thro' the opposite points of division, lines parallel to each other, and they will divide the given line AB into five equal parts, as was requir'd. Fig. 12.

If you will use the *Sector*, apply the length of the given line AB on the *Line of equal parts*, to a number on both sides which is divisible by 5, since it is to divide the line AB into 5 equal parts, as from 200 to 200, the fifth part of which is 40; and the *Sector* remaining thus open, take off the same *Line of equal parts* the distance from 40 to 40, which will be the fifth part of the given line AB. We shall shew in *Prop.* 1. *l.* 1. of *Euclid's Elements*, another way of dividing a given line into equal parts.

PROBLEM XV.

To make a Scale for to lay down Plans withal.

Plate 2. Fig. 13. HAving drawn the two indefinite lines AB, BC, making at the point B any angle whatever ABC, run over on the line BC as many equal parts as you please, and of what length you will, as for example five from B to C. Make as many on the line AB, from B to E, and again as many on the line CD, which ought to be drawn thro' the point C, parallel to the line AB, from C into F, and join all the points of division that are opposite and equally distant from the line BC, by as many right lines, which will be parallel to each other, and to the line BC, and will divide the Parallelogram BCFE into as many other little Parallelograms, all whose Diagonals must be drawn the same way, which then will be parallel to each other.

It is not necessary that the number of divisions in the line BE, shou'd be equal to the number of the divisions in the line BC, for they may be more or less; but they ought to be equal to the number of equal parts in the opposite and parallel line CF, whose length is consequently equal to that of BE, and each ought to be run over as often as you will in a right line, as CF, three times, for example, at the points H, K, D, and BE also three times at the points G, I, A, which must be join'd to their opposites H, K, D, by the parallel lines, GH, IK, AD, the last of which AD, ought to be divided into as many equal parts as its equal and opposite parallel BC, that is to say, the same equal parts that have been run over on the line BC, ought to be run over on the line AD; then draw right and parallel lines thro' the points that are opposite, and equally Distant from the two parallels AB, CD, and the Scale will be finish'd: To which annex numbers from 25 to 25 on the Parallels AB, CD, to signify that each of the parts EG, EB, GI, and AI, comprehend 25 equal parts; which number 25 is found by multiplying the number of equal parts in the line BE, by the number of equal parts in the line BC, so that each Diagonal is found divided into as many equal parts as the line BC, as here into 5, at points, thro' which if you draw as many lines parallel to the line BC, they will divide each of the equal parts of the line BE, also into five less equal parts,

parts, which are found on the great lines parallel to the line AB, namely *one* on the first parallel 1, 1, from the line EF to the next Diagonal; *two* on the second Parallel 2, 2, between the same Line EF and the first Diagonal, that is to say, between the two points 6, 7; so of others. From whence it follows, that the line 8, 7, contains 27 equal parts, the line 9, 7, comprehends 52, which represent Feet, Fathoms, or any other measure you will. Plate 2. Fig. 13.

This Scale thus made, is call'd *Plain Scale*, because it is free to take divisions of what bigness you will, since its length is not determined: But when its length is given, as also the number of its equal parts, it is call'd *Forc'd Scale*, which will not be found difficult to make, to him who understands the Construction of the preceding one; for if the length AB is determined, and of a determinate number of parts, as for example, of 100 Fathoms, because this number 100 is divisible by 4, divide the length AB into 4 equal parts, at the points B, G, I, each of which will represent 25 Fathoms; and because this number 25 is divisible by 5, divide the part EF into 5 equal parts, each of which will represent 5 Fathoms, because by dividing 25 by 5, the Quotient is 5; wherefore to have a Fathom, draw at pleasure thro' the extremity B, the indeterminate line BC, in order to run over 5 equal parts of any bigness from B to C, then the rest may be done as before.

You may upon this principle, make such a Scale several ways, as in *Fig.* 14, which is a Scale of 20 equal parts, and in *Fig.* 15, where you have a Scale of 70 equal parts, which may be taken for Fathoms, Feet, Inches, or for any other Measure you will. You need only look upon these three Figures to comprehend them, and therefore I shall say no more of them; except that if in *Fig.* 13. you run over on the line BC 6 equal parts, each division of the line EB would be taken for a Fathom, and the subdivision had represented Feet, because a Fathom contains 6 Feet, so that the line 6, 7, would have represented two Feet, and the line 8, 7, had represented 5 Fathoms and 2 Feet; and lastly, the whole line AB had been 20 Fathoms.

PRO-

PROBLEM XVI.

To lay down an accessible Plan.

Plate 2. Fig. 16.

FIrst, if you enter within the accessible Place, suppose ABCDE, your best way is to take a foul draught of it on Paper any how, to set down the length in Feet, Fathoms, &c. of each Side, which we will suppose of as many Fathoms as you see mark'd in the Figure, as also the Diagonals AD, BD, which you are at liberty to draw as you will, from one Angle to another, so that the given Plan be reduc'd into Triangles, which must be protracted one after another, by taking from a Scale as many equal parts as each line contain'd Fathoms on the Ground, for thus the whole Figure will be reduc'd into a small compass upon Paper, and the Plan thereof laid down.

But to come to the Practice, draw on Paper the line *ab* of 20 parts taken from the Scale, for the 20 Fathoms of the Side AB; then from the point *b*, with the distance of 25 parts, for the 25 Fathoms of the side BD, of the Triangle ABD, describe an Arc, and another from the point *a* with the distance of 27 parts, for the 27 Fathoms of the other side AD, of the same Triangle ABD, and thro' the intersection *d* of these two Arcs, draw from the two points *a*, *b*, the right lines *ad*, *bd*, which will make with the first *ab*, the Triangle *abd*, similar to the great one ABD, which in this manner is protracted. And thus the two other Triangles BCD, AED, may be protracted; so you have the small Figure *abcde* similar to the great one ABCDE.

If the given Plan be bounded by some Curve-lines, take those Curve-lines for right ones, when they differ but little; otherwise they must be reduc'd into lines insensibly differing from right ones, by drawing several little right lines that will nearly form the Figure, and reduce it into Triangles by drawing Diagonals, then will these Triangles be protracted, and consequently the given Figure, as was just now taught.

Secondly, if it be impossible to get within the given Figure, so as to measure the Diagonals, as if the given Plan was included between Walls, or if it be a Wood, Fenny

place,

place, or a Pond; measure this Plan from without, by taking as before the Sides with a Cord or Chain, and the Angles with an Instrument, as was taught in *Prob.* 8. Then protract it on Paper, by taking its Sides off a Scale of equal parts, and setting down the Angles observed with a Protractor, or otherwise as was taught in *Prob.* 9. And thus the two Figures, *viz.* the great on the Ground, and the little on Paper, will be similar, because of the equality of their Angles, and the proportion of their Sides. Plate 3. Fig. 17.

But since it is easy to mistake, as well in taking the Angles on the Ground, as in laying them down upon Paper, and that a little error with respect of the Angles, occasions a considerable difference; it is better to use the following method, which always succeeded well with me, when I took a little care to produce the Sides in a right line.

Let us propose then the Plan ABCDE, which is accessible without, but does not hinder but you may measure its Sides, which we will suppose of as many Feet as are mark'd in the Figure; Produce one of the Sides AB, to F, as much in a right line as is possible, so that BF be of a certain known length, more or less, according to the conveniency of the Ground, as for example 80 Feet, taking rather Feet than Fathoms, because the Sides of the Plan have been measur'd in Feet; then measure the line FC, and suppose it 70 Feet, which ought to be so done, because this line makes with the other two BF, BC, the Triangle BFC, this being protracted by the means of some particular Scale, which may be supplied by the Sector, taking off the measures on the two Lines of equal parts on both sides, the Sector being more or less open, as you would have the Figure on the Paper to be great or small, then you'll have the position of the Side BC, which cannot be done otherwise, without knowing the Angle ABC, where it is more difficult to succeed well.

Produce in the same manner the Side BC to G, so that CG be of any length, as 50 Feet, and in like manner measure the line GD, which we will suppose 40 Feet, this will give the position of the Side CD, without knowing the Angle BCD; and since there remains no more than the two Sides AE, DE, you may stop there, because that will be sufficient to lay down this Plan on Paper, which is done thus.

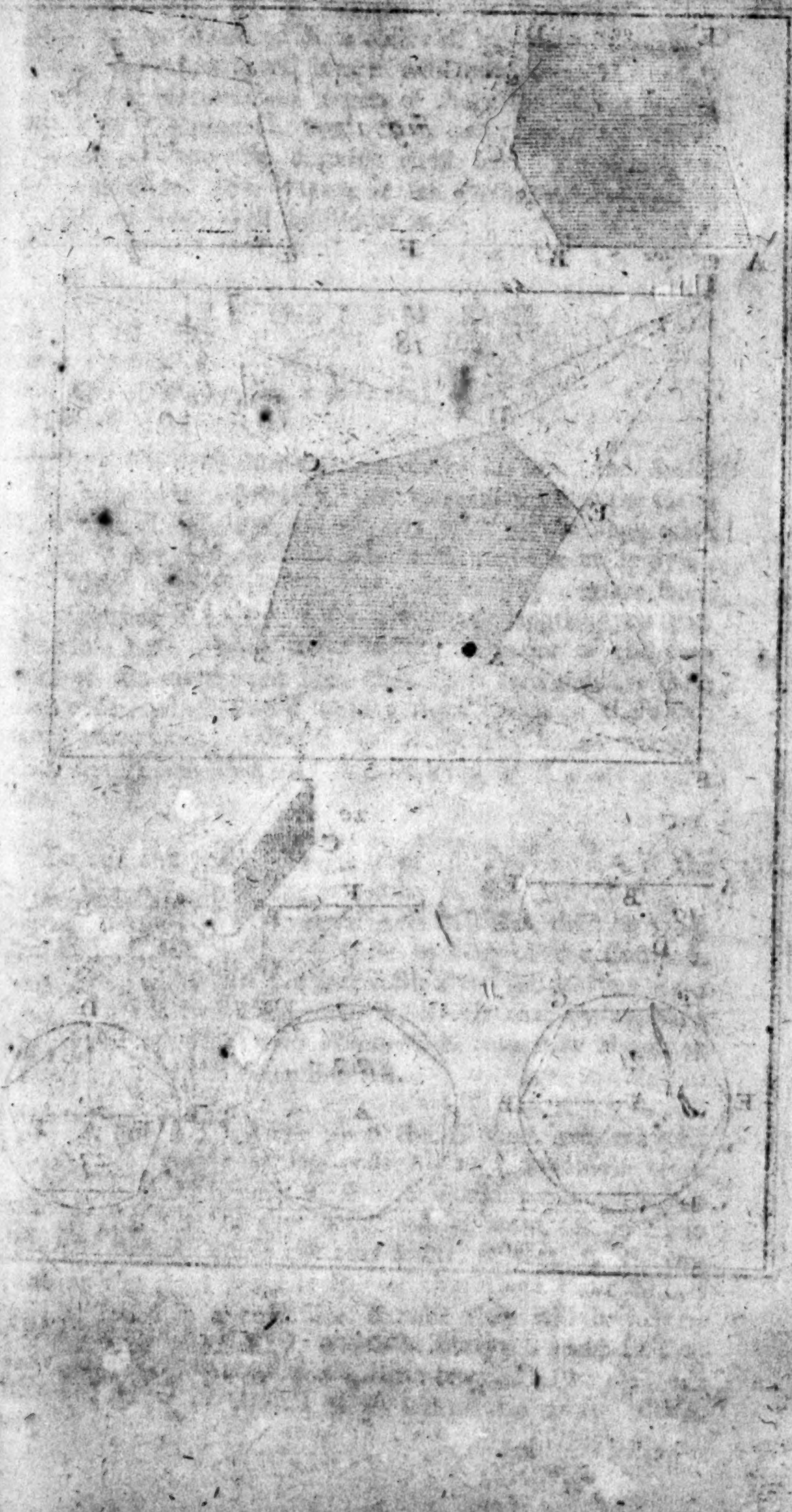
Plate 3. Fig. 17. Having drawn the line *ab*, let it be 100 parts from some Scale, representing the 100 Feet of the great Side AB, and having produc'd it to *f*, so that *bf* be 80 of the same parts, for the 80 Feet of the line BF; from the point *f*, with the distance 70 parts, for the 70 Feet of the line FC, describe an Arc, and another from the point *b* with the distance 60 parts, for the 60 Feet of the Side BC, and thro' the intersection *c* of these two Arcs, draw from the point *b* the Side *bc*, which produce to *g*, so that *cg* be 50 parts, for the 50 Feet of the line CG, and describe as before an Arc from the point *g* with the distance 40 parts, for the 40 Feet of the line GD, and another from the point *c* with the distance 65 parts, for the 65 Feet of the Side CD, and thro' the Intersection *d* of these two Arcs draw from the point *c* the Side *cd*. Lastly, describe an Arc from the point *d* with the distance 90 parts, for the 90 Feet of the Side DE, and another from the point *a* with the distance 100 parts, for the 100 Feet of the last Side AD, and thro' the Intersection *a* of these two Arcs, draw from the two points *a*, *d*, the two Sides *ae*, *de*, and the little Figure *abcde*, will be similar to the great one ABCDE. *See Prob. 5. Chap. 2. Part 3. Geom.*

PROBLEM XVII.

To measure an inaccessible Plan.

Fig. 18. If the Plan ABCDE be inaccessible, so that you cannot measure the length of its sides with a Chain, much less produce them without, nor take its Angles; in such case you must go quite round, describing as you go the Figure FGHI, as near to the place as may be, and as regular as possible, so that the Angles of the given Plan, which are seen from one of the Angles of the circumscrib'd Figure, may also be seen from another Angle of the same Figure, as here the Angle A is seen from the two Angles F, G, as well as the Angle B; the Angle C is seen from the two Angles G, H, and likewise from H, I, which also has in view the Angle D; and lastly, the Angle E seen from the two Angles F, I.

This being suppos'd, measure with the Chain the sides of the Figure FGHI, and with an Instrument take the visual Angles which are form'd at the Points, F, G, H, I; then you



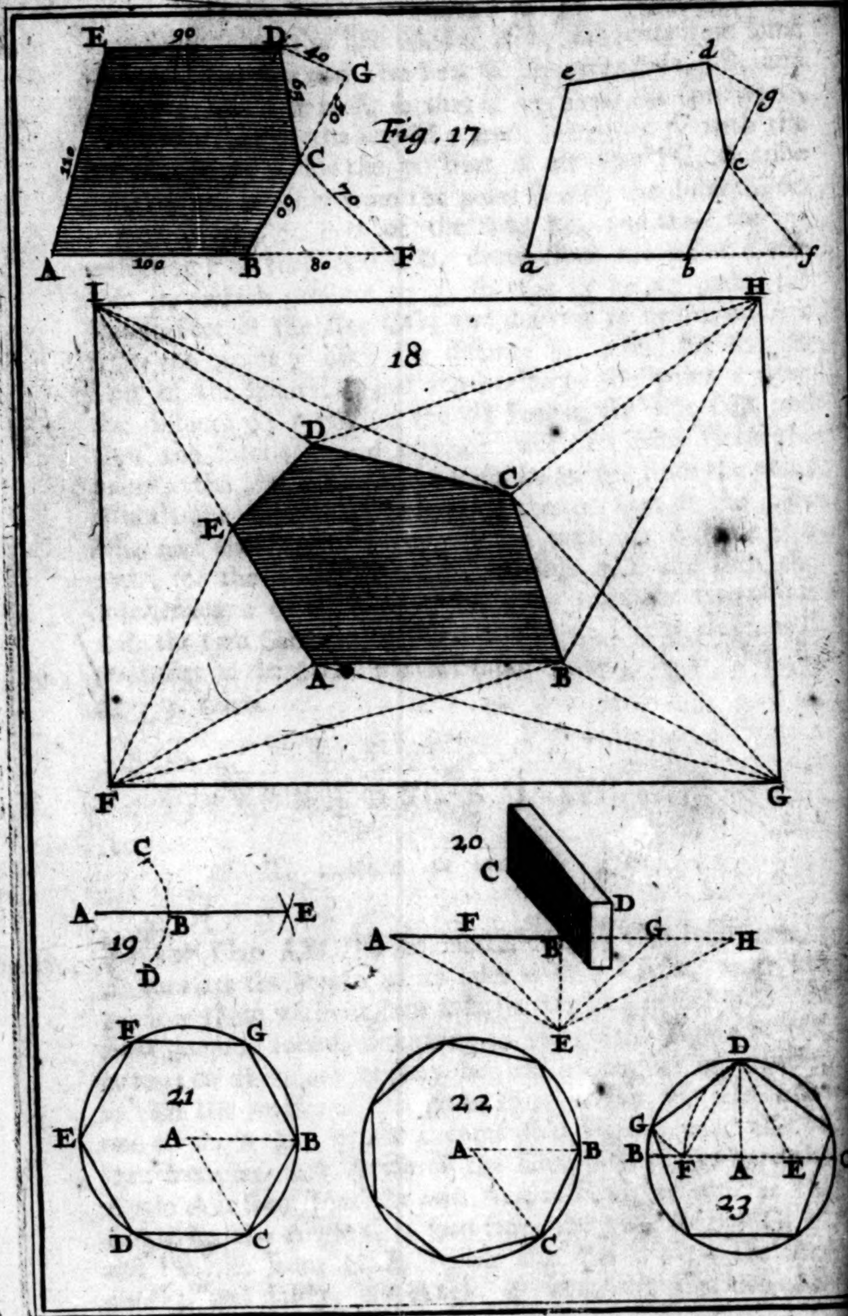


Fig. 17
E
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23

you need only deſcribe upon Paper a ſmall Figure, ſimilar to the great one FGHI, and at the Angles F, G, H, I, make other Angles equal to thoſe obſerv'd, by right lines repreſenting the viſual Rays, which will interſect each other in Points that repreſent the Angles of the given Plan ABCDE, which by this means is Protracted, and reduc'd to a ſmall compaſs on Paper, by drawing right lines from the points of interſection. The Figure it ſelf explains it ſufficiently, ſo that no more need be ſaid of it. Plate 3. Plate 18.

PROBLEM XVIII.

To Produce a Line that is too ſhort.

THO' this Problem be naturally known, and *Euclid* takes it for a Principle, yet in practice, when the given Line is ſmall, it is difficult to do it well by the application of the Ruler, becauſe if you fail ever ſo little in applying the Ruler upon a ſmall extent, you ſenſibly deviate from the right line in an extent of a conſiderable length; you muſt therefore have a point more remote from one of the two ends of the given right line, than theſe two ends are from each other, which ſhou'd be in a right line with theſe two ſame extremities, in order to apply the Ruler thereto, that the given line may be produced with more exactneſs.

To find this point, deſcribe from the extremity A of the given line AB, thro' the extremity B, the Arc CBD, and take at pleaſure the two equal Arcs BC, BD, deſcribe from the ends C, D, with the ſame opening of the Compaſs, two Arcs, whoſe point of interſection E, will be in a right line with the two extremities A, B, for that by applying the Ruler upon the two Points A, E, you may the more exactly produce the given line AB. Fig. 19.

If the line AB is given upon the Ground, you may fix two Stakes upright at the ends A, B, and cauſe a third Stake to be fix'd beyond B, if you would produce the line AB on that ſide to any conſiderable diſtance, as in E, ſo that by looking along the two Stakes fix'd at A, B, you perceive the third Stake in E, for thus theſe three Stakes will be found in a right line, becauſe they will be in one and the ſame viſual Ray, which is always a right line, at leaſt when it is not of too great a length.

Plate 3. Fig. 20. You cannot proceed in the same manner when there's any Impediment, like that of the Wall CD, in this case: And therefore at the point B, let BE be drawn at right angles to AB, and of any length, and draw from its extremity E, thro' the two points A, F, taken at pleasure in the line AB, the right lines EA, EF, measure the Angles BEF, BEA, and the lines EF, EA: Then make on the other side the Angle BEG equal to the Angle BEF, the line EG will be equal to the line BF; make also the Angle BEH, equal to the Angle BEA, and the line EH will be equal to the line EA, then the given line AB may be continu'd beyond the Wall CD, by joining the two points G, H, by a right line, &c.

PROBLEM XIX.

To inscribe a Regular Polygon in a given Circle.

Fig. 21. First, if you would describe an Hexagon in the given Circle BCDEFG, whose Centre is A; the Radius AB being set off on the Circumference, will go round six times exactly, and so give the side of the Hexagon.

Fig. 22. But if you would describe some other regular Polygon, for example an Heptagon, you must on the Centre A make the Angle BAC, equal to the Angle at the Centre, which in the Heptagon is 51 degrees, and about 26 minutes, and the Chord BC will be the side of the Heptagon.

The Angle at the Centre of a regular Polygon is found by dividing 360 degrees by the number of sides of the Polygon, as by 7 for a Heptagon, 8 for an Octagon, and so on.

If you have a *Sector*, apply the length of the Radius AB from 6 to 6, upon the Line of Polygons, and the *Sector* standing thus open, take on the same Line of Polygons, on both sides, the distance from 7 to 7 for an Heptagon, 8 to 8 for an Octagon, and so on, and this distance will be the side of the Polygon sought. See the Treatise we have publish'd concerning *the Use of the Sector.*

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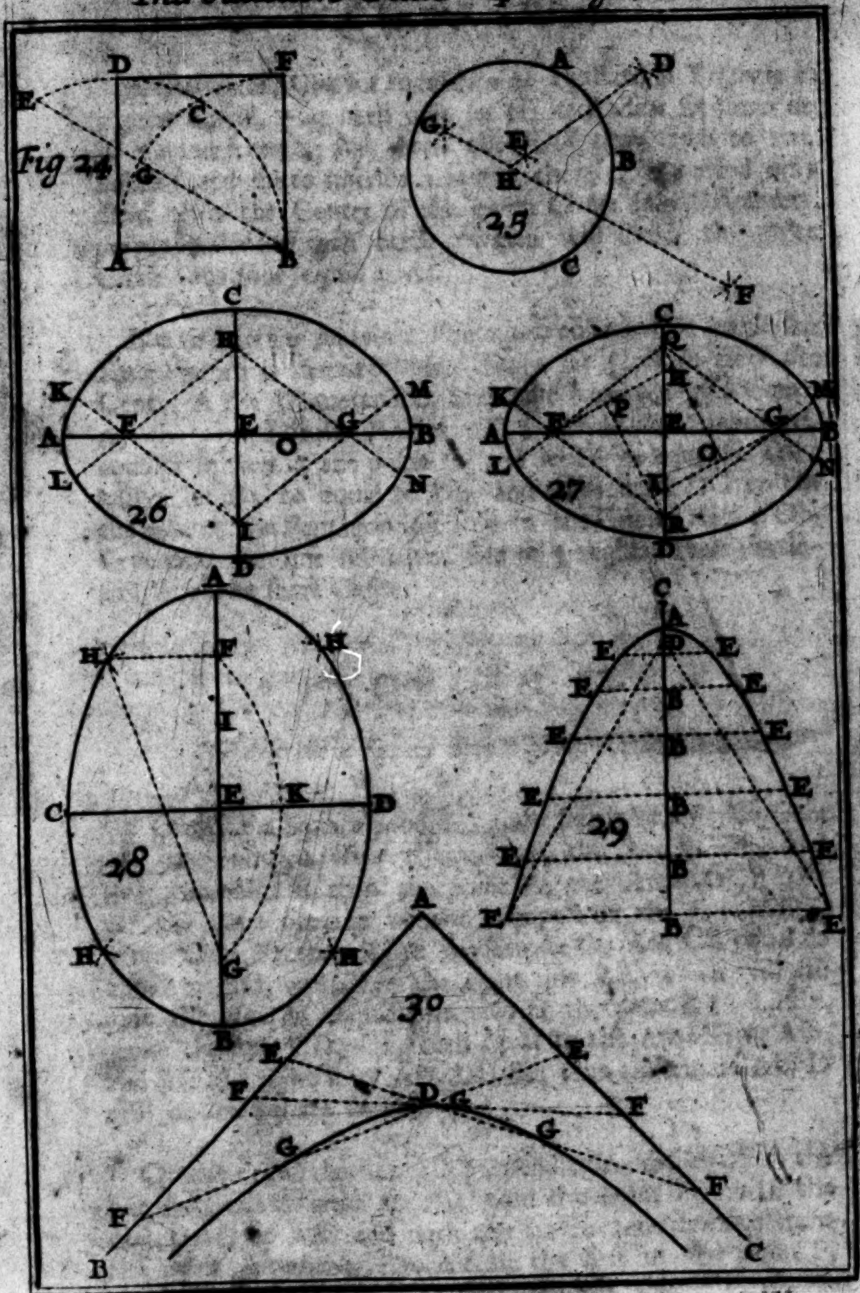

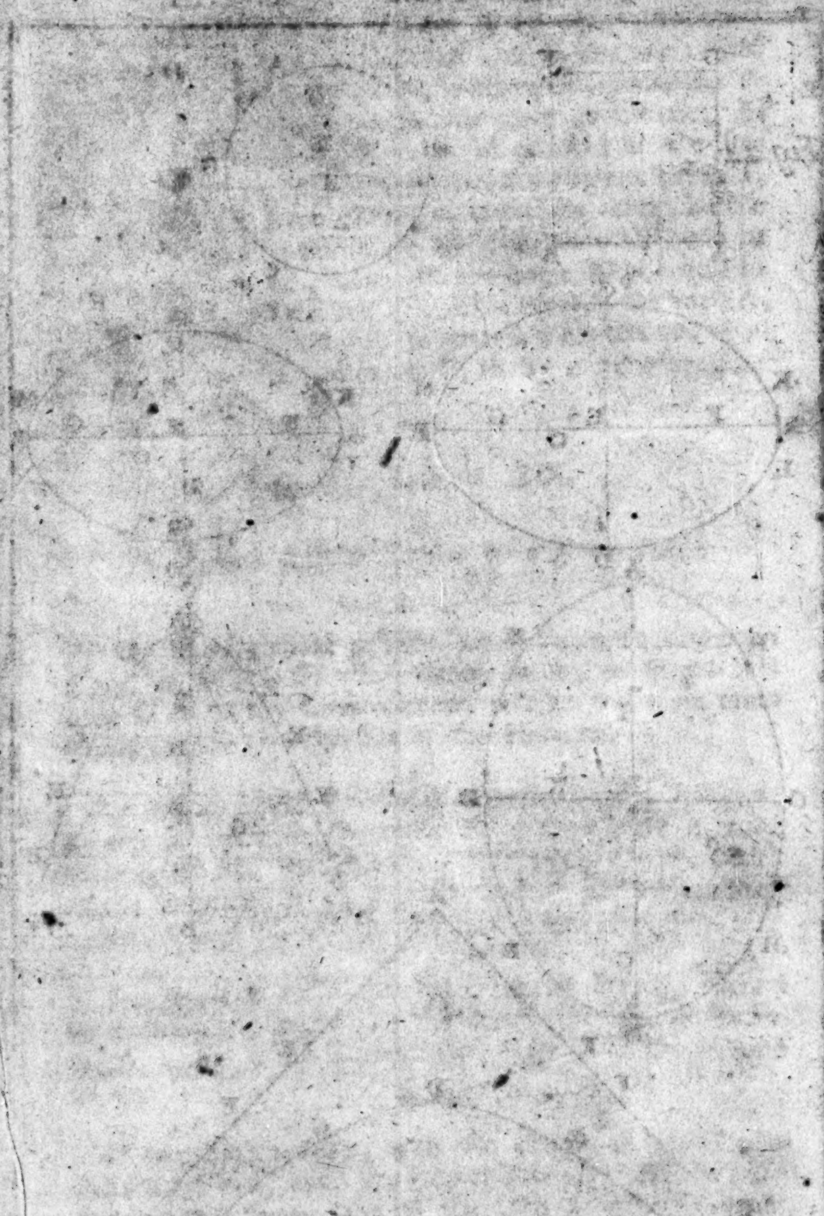

SCHOLIUM.

It is evident, that for to inſcribe an equilateral Triangle in a given Circle, you need only ſet off its Radius ſix times on its Circumference, and draw the ſides from two to two points; and for to inſcribe a ſquare therein, you need only draw thro' the Centre of the given Circle two Diameters, perpendicular to each other, which will divide the given Circle into four equal parts.

But to inſcribe therein a Pentagon, follow this particular Rule, which is demonſtrable. Draw at pleaſure thro' the Centre A the Diameter BC, and raiſe from the ſame Centre A, the perpendicular Radius AD; divide the Radius AC equally in two at the point E, and let EF be equal to DE; laſtly, let DG be equal to DF, and this Chord DG will be the ſide of the Pentagon inſcrib'd in the Circle DGC: Obſerve that the line AF is the ſide of a regular Decagon inſcrib'd in the ſame Circle. Plate 3. Fig. 23.

PROBLEM XX.

To deſcribe a Square upon a given Right-Line.

TO make a Square upon the given line AB; deſcribe from the point A thro' the point B, the Arc BCDE, and from the point B thro' the point A, the Arc AGCF; ſet off the ſame opening of the Compaſs on the Arc BCDE, from C to E, that is to ſay, make the Arc CE equal to the Arc BC, and draw the right line BE, which will divide the Arc AC equally in two at the point G. Laſtly, make the Arcs CD, CF, each equal to the Arc CG or AG, and join the right lines AD, DF, BF, then the Figure ABFD will be the Square ſought. Plate 4. Fig. 24.

Or elſe draw the line AD perpendicular and equal to the line AB, and deſcribe an Arc from the point D, with the extent AD or AB, and with the ſame extent deſcribe from the point B another Arc cutting the firſt in the point F, thro' which draw the right lines FB, FD, &c.

PROBLEM XXI.

To describe a regular Polygon upon a given Right-Line.

Plate 3. Fig. 22.

TO describe upon the given line BC a regular Polygon, for example an Heptagon; make at the two ends B, C, of the line BC, the Angles BCA, CBA, each equal to the half of the internal Angle of the Polygon, which in this instance is 64 degrees 17 minutes, and from the point A where the two equal lines AB, AC, meet, describe thro' the two points B, C, the Circumference of a Circle, wherein may be inscrib'd a regular Heptagon, each side whereof will be equal to the given line BC.

The internal Angle of a Polygon is found by substracting from 180 degrees the Angle at the Centre, which is found by what has been shewn in the foregoing Problem: Or without knowing the Angle at the Centre, by multiplying 180 degrees by the number of sides of the Polygon except two, namely by five for an Heptagon, six for an Octagon, and so on, and by dividing the Product by the number of the sides of the Polygon.

If you have a *Sector*, apply the length of the given line BC upon the Line of Polygons, to a number on both sides equal to the number of sides of the Polygon to be describ'd, as in this case from 7 to 7; and the *Sector* remaining thus open, take with a Compass the distance from 6 to 6 on the same Line of Polygons, and describe with this opening from the two ends B, C, of the given line BC, two Arcs, whose Intersection will give the Centre A of a Circle, in which may be inscrib'd the Polygon proposed, as here a regular Heptagon, where the given line BC will be one of its sides.

PROBLEM XXII.

To describe the Circumference of a Circle thro' three given Points upon a Plane.

Plate 4. Fig. 25.

THE three given points must not lye in a right line, for then the Problem would be impossible. To describe therefore a Circle thro' the three given points A, B, C, which are not in a right line, describe from the two points A, B,

A, B, both ways with the same opening of the Compass, two Arcs, and thro' their intersecting points E, D, draw the indefinite right line DEH. Describe likewise from the two points B, C, both ways with one opening of the Compass, two Arcs, which in this case will intersect in the two points F, G, thro' which draw the right line FG, which being produc'd if occasion requires, will cut the first line DE, in like manner produc'd, in a point, as H, which will be the Centre of a Circle, whose Circumference will pass thro' the three given points A, B, C. Plate 4. Fig. 25.

SCHOLIUM.

By this method a segment of a Circle may be compleated, to wit, by taking at discretion three points in this Arc, and finding the Centre of a Circle which passes thro' these there points.

PROBLEM XXIII.

To describe the common Oval on two given Diameters.

TO describe the common Oval about the two given Diameters AB, CD, which cut each other at right angles and into two equal parts at the point E, which is the Centre of the Oval; set off the length of the little Diameter CD, upon the great one AB, from A to O, and take on the same great Diameter AB, the lines EF, EG, equal to BO, and upon the little Diameter CD, the lines EH, EI, each equal to three fourths of BO, that is to say of EF, or EG. Then draw from the Points H, I, thro' the points F, G, the indefinite right lines IK, IM, HL, HN, which will be terminated at the points K, L, M, N, by describing from the point F thro' the point A the Arc KAL, and from the point G thro' the point B the Arc MBN. Lastly, describe from the point H thro' the two points L, N, the Arc LDN, which will pass thro' the point D; and from the point I thro' the points K, M, the Arc KCM, which will pass thro' the point C; and you will have the perfect Oval ACBD. Fig. 2?.

The like Oval may also be describ'd very easily thus: Take upon the two given Diameters AB, CD, the equal Lines AF, BG, CH, DI, of any length, and join the right lines, FH, GI, each of which bisect in the points O, P, on which erect the two Perpendiculars OQ, PR, which in this Fig. 27.

this case will cut the Diameter CD, in the points Q, R, thro' which, and the two points F, G, draw the indefinite right lines RK, RM, QL, QN, then the rest is done as before.

PROBLEM XXIV.

To describe the Mathematical Oval about two given Axes.

THE Oval we just now describ'd is call'd *the Common Oval*, to distinguish it from *the Mathematical Oval*, commonly call'd *Ellipsis*, and which has in no wise any part thereof circular, it being form'd by the Section of a Cylinder and a Plane which is not perpendicular to the Axis of the Cylinder, otherwise the Section would be a Circle: Or else by the section of a right Cone and a Plane, cutting the two opposite sides of the Cone, and not parallel to the Base of the Cone, otherwise the section would again be a Circle.

Plate 4. Fig. 28. The curve line ACBD represents the Periphery of an Ellipsis, whose principal property is, that if from two certain points F, G, taken upon the greatest Diameter AB, and equally remote from the Centre E, which are call'd *Focii*, be drawn to any point H, of the Circumference, the right lines FH, GH, their sum FH + GH is equal to the greatest Diameter AB, which is call'd the *Principal Axis*; the lesser Diameter CD, which is perpendicular to it being call'd the *Lesser Axis*; and the point E, where these two Axes cut each other, is call'd the *Centre of the Ellipsis*.

This curve line ACBD not being circular, either in whole or in part, cannot be describ'd Geometrically, but by finding several points Geometrically, and joining them dextrously by one continued curve line, which will determine the Ellipsis; and this will be so much the easier, the more points there are found.

There are several methods for finding out these points; among others I have made choice of the following, which seems to me better than any for practice. Its Origin and Demonstration is drawn from the precedent property of the *Focii* F, G, which are to be found in the great Axis AB, by describing from the extremity C of the little Axis CD, with the extent of the great Semi-axis AE or BE, the

the Arc FHG, which will cut the great Axis AB in the *Focii* F, G, by means of which an infinity of points in the Curve of the Ellipsis may be found, thus. Plate [illegible] Fig. [illegible]

From the *Focii* F, G, with any distance in the Compass greater than AF, or BG, describe small Arcs both ways, and having set off this same distance on the great Axis AB, from A to I, and from the said *Focii*, with an opening of the Compass equal to the Remainder BI of the great Axis AB, describe other Arcs, cutting the former in four points H, which will be the points in the Curve of the Ellipsis. In the same manner, by describing Arcs greater or less, from the the said *Focii* F, G, you will find as many other points in the Ellipsis as you please, which points being join'd by a Curve line, the Ellipsis will be describ'd.

If you have no *Compasses*, you may find as many points of an Ellipsis as you please, by the help of the *Ruler* only, namely by setting off on the edge of the said Ruler from its end, the length of the great and small Semi-axis, which may be done without Compasses, if you apply the end of the Ruler to the Centre E, and the edge of the same Ruler on each of the two Semi-axes EB, EC, and mark upon the same edge the points where the two ends shall terminate; and by applying these two points upon the two Axes AB, CD, so that the point of the small Semi-axis answers on the great Axis AB, and reciprocally the point of the great Semi-axis upon the small Axis CD; for then the same end of the Ruler will note a point in the Ellipsis; and as this application may be made an infinite number of different ways, it is evident that by this means may be found as many different points of the Ellipsis as shall be desir'd.

This Method has its Demonstration, and is the foundation of a certain Instrument not uncommon, and made use of to describe an Ellipsis at once, as the common Compass is made use of to describe a Circle. But there is another very easy way of describing an Ellipsis at once, by a more simple method, depending upon the general property of *Focii*, which we have mention'd already, and is common enough among Artificers.

Having found the two *Focii* F,G, as before shewn, tie thereunto a Cord, whose length must be equal to that of the Ellipsis, *i. e.* to the given great Axis AB, then there needs

no

Plate 4. Fig. 28. no more than to stretch out this Thread or Cord with a Pen or Pencil, which you must move along the said Cord equally extended, and this Pen will by its motion describe the Circumference of an Ellipsis, where the two given lines AB, CD, will be the two Axes thereof, that is to say, the length and breadth. This Cord is represented in the figure by the line FHG.

PROBLEM XXV.

To describe a Parabola on a given Axis.

Fig. 29. THE Parabola is the section of a Cone and a Plane parallel to one of the *Sides of the Cone*, that is to say, to a right line drawn from the Vertex of the Cone thro' some point of the Circumference of its Base, which is a Circle. This Section or Parabola is bounded by a Curve Line call'd a *Parabolical Line*, and generally a *Conic Line*, because a Conic Line is the Section of a Plane and a *Conic Superficies*, that is to say, the Surface of a Cone. It is evident that this Parabolical Line is a Curve Line, and spreads in its progress not unlike a Rope slack pull'd, or a heavy body, which being thrown obliquely into the Air, descends with much the same obliquity, describing a Parabolical Line.

The essential property of the Parabola is, that draw within the Line as many Parallels as you please, such as EE, divided equally in two at the points B, by the right line AB, which in this case is call'd the *Diameter of the Parabola*, and the *Axis*, when it is perpendicular to these Parallels, call'd *Ordinates*, with respect to the Diameter AB, which divide each of them equally in two; the Squares of all these Ordinates, are proportional to the corresponding parts of the Diameter AB, taking them from the extremity A, which is call'd the *Vertex of the Parabola*: From whence may be drawn a Construction of the Parabola, but it will not be so easy as that which is derived from the property of its *Focus* D, which is such a point in the Axis AB, that if upon this Axis AB produc'd, you take the part AC equal to the part AD, the part CB is equal to the corresponding line DE: Which gives a very easy method to find out as many points in the Parabola as shall be desir'd.

To describe therefore a Parabola, thro' the point A of the given Axis AB; Take upon this produc'd Axis AB the

the equal lines AC, AD, great or small, according as you would have your Parabola more or less open. Take on the same Axis AB, below the Vertex A, as many different points as you would find in the Parabola, as B, thro' which draw the indefinite lines EBE perpendicular to the Axis AB, in order to mark out the points E of the Parabola, by setting off the distances CB from the Focus D, on both sides on their respective perpendiculars, *&c.* Plate 4. Fig. 29.

PROBLEM XXVI.

To describe an Hyperbola thro' a Point given between two given Asymptotes.

AN Hyperbola is the Section of a Cone cut by a Plane, which being produced, meets the Cone in like manner produced, without its Vertex, and the Asymptotes are two right lines, as AB, AC, which cut each other in the point A, call'd the *Centre of the Hyperbola*, which Lines being produc'd as much as ever you will, can never cut the Hyperbola GDG, so far off as it is produced, tho' they still approach nearer to it, they being always distant from it by a less quantity than any other that can possibly be conceiv'd. Fig. 30.

The property of these Asymptotes is such, that if you draw within their Angle a right line at pleasure, as EF, which cuts the Asymptotes in the two points E, F, and the Hyperbola in the two points D, G, the lines DE, FG, are equal to each other. And therefore if the point D be given within the Asymptotes AB, AC, thro' which point an Hyperbola is requir'd to be describ'd, draw thro' the said given point D any right line as EF, upon which set off the length of the part DE, bounded by the given point D, and one of the Asymptotes, beginning at the point F, from the other Asymptote to the point G, which will be the point of the Hyperbola, *&c.*

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THE
Elements of *Euclid*
Explain'd and Demonstrated in a short and easy Method; with the Use of the Propositions.

ALTHO' our Design in this short Treatise (or *Course of the Mathematicks*) is not to explain all the Books of *Euclid*'s Elements, but only the *Six first*, the *Eleventh*, and *Twelfth*, (which will be sufficient for the understanding all the rest we shall here offer afterwards); We shall, notwithstanding, follow *Euclid* Step by Step, without in the least receding from his Method *of supposing nothing but what has been before-hand, either laid down by way of Principle, or else demonstrated*; without changing any thing in his Method or Constructions, when they are at the same time both general and easy, and depend upon some Proposition or Propositions that have been before demonstrated; that so we may give every Proposition its just Value and Use, which some have neglected to do, and that particularly when in following *Euclid*'s Method the Solution had been more universal. Thus (for Example) after *Euclid* has taught us *to construct a Triangle of any three Lines given*, for a Man to 22. 1.
have recourse to solve the following Problem, *viz. To make* 23. 1.
an Angle at any Point of a given Line, equal to an Angle given; this would be impertinent, and beside the Author's Intention, as well as contrary to the Order and Beauty of a methodical Process in these Sciences. To resolve this last Problem without making use of the Precedent, is neither so general nor Geometrical. However, to give the Reader as little trouble as possible, and abridge our Work, we shall imitate F. *Tacquet*, or *Dechales*, in not troubling the Reader with those Propositions we shall think unnecessary and of no consequence, or of but little Use to demonstrate those that follow: We shall also endeavour to illustrate the principal Propositions by the most familiar Examples we possibly can. Those

that desire any more may consult *Henrion*; who is the best Commentator upon *Euclid* I know.

The FIRST BOOK of *EUCLID*'s ELEMENTS.

EUCLID treats in this First Book, of Lines, of Angles, and of Triangles, and other right-lin'd plane Figures, and chiefly Parallelograms, shewing the Method of reducing any right-lin'd Plane into a Parallelogram, in order afterwards to reduce it (or make it) into a Square, as he shews in his Second Book; at the end of which, he demonstrates that celebrated Proposition of *Pythagoras*, That in a *right-angled Triangle*, the Square of the greatest of the Sides (commonly call'd the Base, or Hypothenuse) is equal to the Sum of the Squares of the other two: which is the Foundation of Geometrical Addition, and Substraction too, in the Case of adding or substracting of Planes; *i. e.* whereby several Planes may be summ'd up (*viz.* their Area's) into one, and consequently one found equal to their Sum.

DEFINITIONS.

I.

A Mathematical Point *is that which has no Parts* (or at least is what is consider'd as such) *and which of course is indivisible; and which consequently has no other Existence, than in the Understanding of those that think of it.*

By this Definition, a *Mathematical Point* may be distinguish'd from a *Physical one*, which may be perceiv'd by our Senses, as having Parts. Yet notwithstanding that, we often use them promiscuously, the one with the other, upon the score we never consider it (when we think of it as such) as capable of being subdivided: Thus when we say a certain thing is exactly so many Feet long, we consider the Yard or Foot as an whole (or undivided Quantity) and consequently as an indivisible Point, that is, as a Mathematical Point: But yet if besides the determinate Number of Feet, there should happen to be some odd Inches, then the Inch would be consider'd as the Indivisible (or Mathematical) Point, as being the least Subdivision; which, as such, would be taken for a Physical Point. II. A

II.

A Line *is a Length without either Breadth or Thickneſs, which of courſe can only be an Object of the Underſtanding.*

We generally ſay, that a Line is generated by the Motion of a Point, whence it can neither have Length nor Breadth, and may be conceived as the Motion or Flux of a Point from any one determinate Part of Space to another; or, as we cannot poſſibly trace out any Line *(in matter)* whatſoever, that is not a Phyſical one, or which, beſides its Length, has not ſome Breadth and Thickneſs; yet that will be no Obſtacle but that we may conceive or take it for a *Mathematical Line*, while we only conceive it as Length; as when we only conceive the Length of a Journey, without making any Reflections on the Breadth, *&c.* of the Way.

III.

The two Extremities *(or Ends)* of a Line *are Points.*

This is to be underſtood of thoſe Lines only that have two *Extremities* (or Ends); nor does it hence follow that all Lines have two Ends; it being certain that thoſe which include, or every ways terminate, Space, ſuch as the Circumference of a Circle, an Ellipſe, *&c.* have no Ends.

IV.

A Right-Line *is that whereof all the Points are equally plac'd between its two Extremities.*

Whence it follows, that a *Curve-Line* is that which has not all its Points plac'd equally between its two Extremities, becauſe ſome are elevated above, and ſome ſubſide below others.

V.

A Superficies *or* Surface *is an Extenſion, or Space extended, without any Thickneſs or Depth.*

As a Line is the firſt Species of continued Quantity, having but one Dimenſion, *viz. Length*, ſo a *Superficies* is a ſecond Species of it, becauſe it has two Dimenſions, *viz.* Length and Breadth: And as a Line is conceiv'd to be produc'd by the Motion of a Point, ſo may we conceive a Surface to be produc'd by the Motion of a Line: And finally, as a Line conſiſts of an infinite Number of Points, ſo does a *Surface* conſiſt of an infinite Number of Lines.

VI.

The Extremities *or* Ends *of a Surface, (*viz. *when it has any) are Lines.*

This follows from the Nature of a Surface, which being compos'd of an infinite Number of Lines, muſt needs

be terminated by them, if it be terminated at all: Which is to be thus underſtood then only, when both the one and the other of theſe two Species of Quantity have *Extremities* or *Ends*; for we have already taken notice, that the Circle, Ellipſe, *&c.* are terminated by one Line only, which has no End; or to ſpeak more properly, whereof the two Ends are joined together; thus we ſhall in the ſame Senſe take notice, that a *Sphære*, a *Sphæroid*, &c. are terminated by one only Surface, which has no Ends.

VII.

A Plane-Surface, *or a* Plane, *is that which has all its Right-Lines equally plac'd between its Extremities; ſo that one does not riſe higher or ſubſide lower than the other.*

Whence it follows, that a *Curve-Surface* is that which has not all its Parts placed equally between its Extremities, one riſing higher, another falling lower, than each other: And when ſuch a Surface is conſider'd in relation to the Side that ſubſides, it is call'd a *Concave-Surface*; and when it is conſider'd on that which riſes up, it is call'd *Convex*. Thus the Happy above may be conceiv'd to ſee the Convex-Side of Heaven (according to the *Ptolemaick* Syſtem) while thoſe below can only ſee the Concave Part of it.

VIII.

Plate 1, Fig. 1. *A* Plane-Angle *is an indefinite Space terminated by two Lines inclining to one another* [or rather by the meeting of thoſe two Lines] *when they meet in a Point upon the Plane where the Angle is formed, and don't by that meeting make a Right-Line, as* ABC.

Hence you ſee, that to form an Angle, it is not only neceſſary for the two Lines to meet at the *angular Point*, but to meet likewiſe in ſuch manner, as that being produc'd, they ſhall interſect, and afterwards deviate from each other.

You alſo ſee, that the Magni[illegible] of the Angle does not depend on the Length of the Lines that form it, but on the Quantity of the Inclination; for it is evident from the Definition, that the more or leſs the Lines are inclin'd, the Angle will alſo be the greater or the leſs: And the *Angle* is denominated *Plane*, becauſe it is deſcrib'd on a Plane. There are three Sorts of them, which we ſhall now explain.

IX.

A right-lined Angle *is that whereof the two Lines that form it are Right-Lines; as in ABC, the two Lines BA, BC, are Right-Lines; as alſo in the Angle ABK, where BA and BK are Right-Lines.*

It is

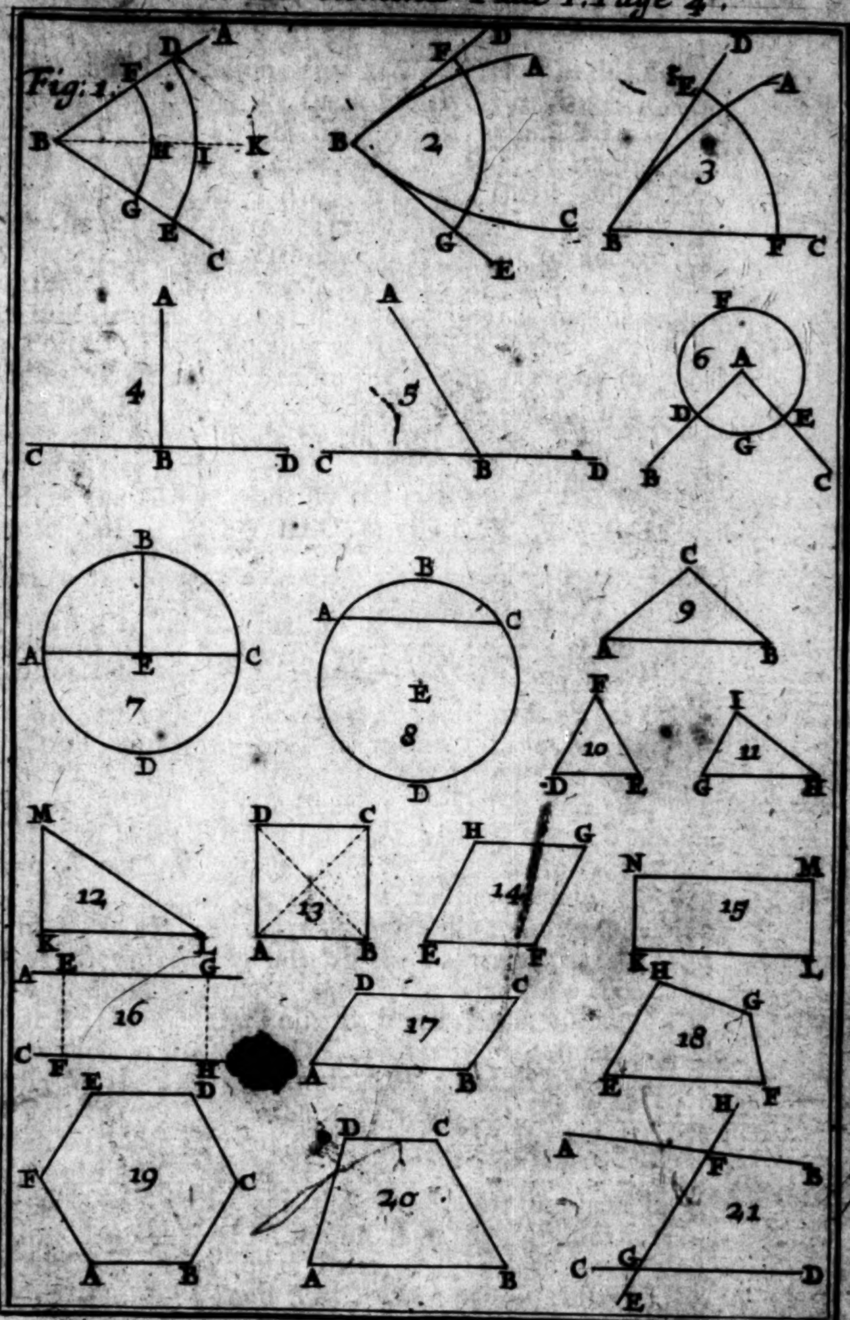
Fig: 1.
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It is this Angle alone that *Euclid* treats of in this Book, wherefore whenever we ſpeak ſimply of an Angle, it is to be underſtood of a right-lin'd Angle, which may be denoted by one only Letter, *viz.* by that at its angular Point, when one only Angle is formed there; but when at the ſame Point there are more Angles than one, formed by more Lines that terminate there, then to denote the particular Angle we mean, we make uſe of three Letters, the middlemoſt whereof ſignifies or points out the angular Point. Thus, becauſe at the Point B there are three Angles, if we would denote the Angle made by the two Lines BA, BC, we ſhould write it thus, ABC; and if we meant the Angle made by the two Lines, BA, BK, we ſhould write it thus, ABK; and in like manner to repreſent the Angle made by the two Lines, BK, BC, we ſhould call it either KBC, or elſe CBK; and ſo of others.

We have already ſaid, that an Angle is greater or leſs according as the Inclination of the Lines that form it is greater or leſs: And here we ſhall acquaint the Reader, that the meaſure of a right-lin'd Angle is determin'd by the Arch of a Circle deſcrib'd at pleaſure from its angular Point, and terminated by the two Lines of that Angle: Thus the meaſure of the Angle ABC is the Arch DE, or alſo FG, whoſe Centers are at the Point B; the Arch DE being exactly the ſame part of the Circumference of its Circle, as the Arch FG is of its reſpectively: For if you imagine the Line BC to move about the fixt Point B, ſo that it may make with the immoveable Line AB Angles greater or leſs, all the Points of the ſaid Line BC will move circularly, and at the ſame time about the Point B. So that the Point E, for example, will deſcribe by its Motion the Arch DE, which by conſequence will be the Meaſure of the Angle ABC; and in like manner, the Point G will deſcribe, by its Motion, the Arch FG, which will alſo, by the ſame Reaſon, be the Meaſure of the Angle ABC, and ſo of others.

It will be eaſy to conclude from what we have been ſaying, that the Right-Line BK ſhall then divide the Angle ABC into two equal Parts, that is into two equal Angles, *viz.* ABK, and KBC, when paſſing through the Point B, it ſhall divide DE, the Meaſure of the Angle ABC into two equal Parts in the Point I, that is into two equal Arches, ID, and IE, which are the meaſures of the equal Angles ABK, KBC. Where we ſee that two Angles, as ABK, KBC, are equal, when their Meaſures ID, IE, which are deſcrib'd from their angular

Plate I. Points with the same Opening of the Compasses, are equal.

Fig. 2. By what we have been saying, it will not be difficult to guess at what will be the Measures of a *Curve-lined Angle*, which is a Plane-Angle contain'd under two Curve-Lines, as ABC; for you are only to compare the curve-lin'd Angle ABC, with right-lin'd one DBE, whereof the right-lines DB, DE, touch at the Point B, the two Curve-Lines AB, AC, the Inclination whereof can never so little change, but the Aperture of the Lines that touch them must change also at the same time: For which Reason, if from the Point B, you describe at pleasure the Arch of the Circle FG; that Arch, *viz.* FG, which is comprehended under BD and BE, being the Measure of the right-lined Angle BDE, shall also be the Measure of the curve-lined one ABC.

Fig. 3. After the same way we also may determine the Measure of a *Mixt-lined Angle*, or an Angle comprehended under a Curve-Line and a Right-Line, as ABC, *viz.* by drawing thro' the Point B, the Right-Line BD, which shall touch the Curve AB in B; and by describing at pleasure from the same Point B, the Circumference of a Circle, the Part whereof FE, comprehended under the Right-Line BC, and the Tangent BD, shall be the Measure of the mixt-lin'd Angle ABC.

It evidently follows from what has been said, that when two Lines only touch one another, they cannot form an Angle, [that may be compar'd with a Right-lin'd one] because they are not inclin'd the one to the other. Thus the imaginary *Angle of Contact*, made of the Tangent and Circumference of a Circle, is improperly call'd an Angle. We have made this Remark upon it, in our Notes we have elsewhere made on the *Euclid* of F. *Dechales*.

'Because that which is call'd the Angle of Contact is 'less than any right-lin'd Angle whatsoever, it follows 'that it is equal to nothing, or that it is nothing. Thus 'we see, that when a Right-Line touches the Circumference of a Circle, it does not make an Angle with it. 'Wherefore the Difficulties that arise from it will vanish, 'when we consider that that Contact does not make an 'Angle, as they only arise from the Supposition that it 'does, and that the Definition of an Angle has not been sufficiently cleared up, nor has it been well enough defin'd 'what the *Contact of two Quantities* is.

'Wherefore we say in general the *Contact of two Quantities* is the meeting of those two Quantities so, that being

'ing produc'd, they shall not intersect one another; that
'is to say, they are not inclined to each other. Whence
'it follows, that an Angle is not rightly defin'd by the
'Contact of two Lines, and that this (whatever it is to be
'call'd) ought to be defin'd from the Meeting of the two
'Lines that compose it; for it does not follow, because
'two Quantities touch one another, that therefore they
'make an Angle; for when those two Quantities are
'Right-Lines, all the Parts of the one coincide with all
'the Parts of the other, when they touch: Whence they,
'not being inclin'd to each other, do not intersect, and
'so make no Angle, tho' they meet and touch. The
'same thing may be said of any Right-Line that touches
'a Curve, because in Contact they are not properly in-
'clin'd to one another, and do not make an Angle. For
'altho' the Curve seem to approach to and recede from
'the Right-Line by its Curvature, and by consequence
'to incline to the Right-Line, and to make an Angle
'with it, that only proceeds from the Figure of the
'Curve-Line, which may be several ways diversify'd,
'and yet make the same Angle with the Right-Line:
'Whence it is easy to conclude, that a Tangent to a Cir-
'cle does not make an Angle with its Periphery. This
'being rightly understood, all the Difficulties that can
'arise upon the Contact of these two Lines, which are
'improperly call'd an Angle, will vanish.

'What I have been discoursing of, may (perhaps) be
'better conceiv'd, if we consider, that an Angle form'd
'by two Curve-Lines, ought to bear some Proportion
'to a right-lined Angle, form'd by the Meeting of two
'Right-Lines, that touch the two Curves in the Point
'where they meet (or in the Point of Contact); because
'according as those two Lines incline to one another
'more or less, the two Tangents should do so also, and
'consequently form a greater or less Angle, which
'would also be the Measure of the Quantity of the
'curv-lin'd Angle. Whence it follows, that when those
'two Curves come to touch one another, they will make
'no Angle at all, because the two Tangents will coin-
'cide.

'Hence it is we say, for example, that if from any
'Point of the Circumference of the Ellipse, we should
'draw two Right-Lines to the two *Focii*; those two Right-
'Lines would make, together with the Circumference,
'two equal Angles; I say those two Angles are not pro-
'perly determined by the Circumference of the Ellipse,
'but by a Right-Line (or Tangent) that is imagin'd to

'fall upon the Circumference without-side, at the Point
'where they make those Angles.

X.

Plate 1. Fig. 4. *When a Right-Line falls upon another, and makes the Angles on both Sides equal, so that it does not incline more to the one Side than the other; each of those Angles is called a Right-Angle, and each of those two Lines is said to be perpendicular to the other. Thus we know that the Line AB is perpendicular to CD, because it makes with that Line CD on each Side, the equal Angles ABC, ABD, which for that reason are called Right ones.*

Those that do not understand the Mathematicks, commonly call a Perpendicular a Plumb-Line, without considering that a Plumb-Line is that Line only which is perpendicular to the Horizon, as a Thread would be with a Lead or Weight hung at the end of it, which we thence call a *Plummet*. Whence, if the Line CD was *horizontal*, or parallel to the Plane of the Horizon, its Perpendicular AB would be a *Plumb-Line*; and if the Line CD was not horizontal, but inclin'd to the Plan of the Horizon, if the Line AB still made with CD equal Angles on both Sids, it would not cease to be perpendicular, tho' it would to be a Plumb-Line, but would be just as different from that, as the Line CD itself would be from being horizontal; and both would become inclin'd to the Horizon.

XI.

Fig. 5. *An Obtuse-Angle is that which is greater than a Right-one; as ABD.*

We may add to this Definition, that the Measure of an Obtuse-Angle is the Arch of a Circle less than a Semicircle, because *Euclid* does not consider any Opening of two Right-Lines that should be measur'd by an Arch greater than a Semicircle, as an Angle, as may be seen in the 21. 3. Thus the Inclination of the two Lines AB,
Fig. 6, AC, makes an Angle at the Point A, that is not measured by the great Arch DFE, which is bigger than a Semicircle; but by the little one DGF, which is less than a Semicircle.

XII.

Fig. 5. *An Acute-Angle is that which is less than a Right-one; as ABC.*

Those two Angles, *viz.* the Acute and Obtuse, differ from a Right-one in this, that there is but one Species of Right Angles, there not being some greater and some less; whereas among Acute and Obtuse-Angles there

many

may be an Infinity of bigger and less, because their Measures may be greater or less Parts of a Circle. It may be easily seen by the Figure, that when one Right-Line falls upon another to which it is not perpendicular, it may in this case be call'd an *Oblique-Line*; which also gives occasion to call an *Oblique-Angle* either an *Acute-Angle*, or an *Obtuse*-one; that is to say, an Angle that is not a Right-one; and it makes on one Side an Acute-Angle, as ABC; on the other an Obtuse-one, as ABD. Plate 1. Fig. 5.

XIII.

The Term *is the Extremity of any thing.*

Hence it is evident there are three Sorts of Terms, *viz.* a *Point*, which is the Extremity of a Line; and a *Line*, which is the Extremity of a Surface; and a *Surface*, which bounds or terminates a Body; which cannot be the Extremity of any other real Quantity, at least that we know of.

XIV.

A Figure *is any Space or Quantity of two or three Dimensions, comprehended under, or bounded every way by, one or more Terms.*

It follows from this Definition, that neither a Line nor an Angle can be called Figures, because a Line tho' bounded by two Points, *viz.* when a Right-Line, and finite, has but one Dimension: And an Angle, tho' bounded by two Lines, yet is not bounded every where, the Space which those two Lines include being indefinite or infinite. Among Figures which are terminated by one only Term, are the Circle, the Ellipse, the Sphere, &c. and among Figures bounded by several Terms, are the Triangle, the Square, the Pyramid, &c. A *Plane-Surface* is called a *Plane-Figure*, or simply a *Plane*.

XV.

A Circle *is a Plane-Figure, terminated by a Boundary of one Line only, which is called its Circumference, as ABCDA, within which is a Point, as E, called its Centre; from which all the Right-Lines EA, EB, EC,* &c. *drawn to the Circumference, are equal to one another.* Fig. 7.

The Vulgar commonly call the Circumference the Circle; as *e.g.* the Hoop of a Tub, abstracting from the *Plane* that is bounded by that Circumference, which notwithstanding is what Mathematicians properly call a Circle, and which nevertheless they themselves too often confound with its Circumference; as *e.g.* when they propose from a given Point to describe a Circle; whereas

they only mean the Circumference of a Circle. In like manner, when they say that two Circles can only intersect or cut one another in two Points, they mean it only of the two Circumferences, as *Euclid* has demonstrated it in the 10. 3.

The Circle might also be very well defin'd a Plane-Surface, produc'd by the Motion of a finite Right-Line moving about a fix'd Point (till the Motion end where it began) which fix'd Point is call'd the Centre, and to which one end of the Right-Line is conceiv'd to be fasten'd, while the other describes by its Motion the Circumference of the Circle.

We commonly say the Circle is the most perfect of all Plane-Figures, because there is no irregularity in it, its Circumference being every where equally round, and its Area the greatest of all Isoperimetrical Figures; *e. g.* its Area is greater than that of a Square of an equal Perimeter.

XVI.

Plate 1. Fig. 7. *The* Centre *therefore of a Circle is a Point within its Circumference, from which all Right-Lines drawn to that Circumference, are equal among themselves; as if E be the Centre, the Lines EA, EB, EC,* &c. *are equal.*

We might also say, that the Centre of a Circle is a Point within its Circumference, placed at the greatest Distance possible from it: Whence we define the Centre of a right-lin'd Figure to be, a Point in the Figure at the greatest Distance possible from its Periphery: Whence it also follows, that the Centre of a regular Polygon, is the same as the Centre of a Circle that circumscribes it; and that the Centre of an Ellipse, is that Point where its two Axes, which determine its greatest Length and Breadth, intersect each other.

XVII.

Fig. 7. *The* Diameter *of a Circle is any Right-Line drawn thro' its Centre, and terminated by the Circumference on each Side; as AC.*

It is hence evident, that a Circle has an infinite Number of different Diameters, which are all equal to one another, and that each divides not only the Circumference, but also the Area of the Circle into two equal Parts.

It is also evident, that a Right-Line drawn from the Centre of a Circle to its Circumference, as EA, EB, EC, is equal to half the Diameter of that Circle, and for that reason is called a *Semidiameter*, as also *Radius* of the Circle.

Fig. 8. And any Part of the Circumference less or greater than its half, is called an *Arch* of that Circumference; as ABC, or ADC.

XVIII. A

XVIII.

A Semicircle *is a plane Figure, terminated by the Diameter of a Circle, and by half its Circumference; as AECBA, or AECDA.* Plate 1.

This Figure is called a Semicircle, because it is equal to half the Circle. Hence also the half of a Semicircle is call'd a *Quadrant*, as AEBA, or DECB, which is terminated by two *Semidiameters* or *Radii*, perpendicular to one another, and by the fourth Part of the Circumference of the Circle, which is sometimes confounded with the Quadrant; as when we say that the Quadrant of a Circle is the Measure of a Right-Angle, instead of saying that the fourth Part or Quarter of the Circumference is so. Fig. 7.

XIX.

The Segment *of a Circle is a Part of a Circle, terminated by a Part of its Circumference, and by a Right-Line; ACBA, or ADCA.* Fig. 8

It is evident by this Definition of *Euclid*, that a Semicircle is a *Segment* of a Circle: But commonly we mean by a *Segment* of a Circle, a Part of it either greater or less than a Semicircle: Whence it follows that the Right-Line that terminates or bounds it, must needs be less than the Diameter, and by consequence can't pass thro' its Centre, as AC, which can't pass thro' E. Here (as I suppose) *Euclid* did not design to leave this Definition thus, because it supposes the Diameter to be the greatest of all Right-Lines that can be drawn within the Circle, which stands in need of a Demonstration, and which is demonstrated in the 15. 3. where *Euclid* repeats the Definition of the Segment of a Circle, it being his Design in that Book to demonstrate its Properties; wherefore he seems only occasionally to have inserted it here.

XX.

A right-lined Figure *is that which is terminated by Right-Lines.*

Whence it follows, that a Curvilined Figure is that which is terminated by Curve-Lines; and a Mixt-Figure that which is terminated by both Right-Lines and Curves. *Euclid* treats here only of right-lined Figures, whereof he shews the Properties of several, which we shall explain in order.

XXI.

A Figure consisting of three Sides (which is also called a Triangle) *is a Figure terminated by three Right-Lines; as ABC.* Fig. 9

A Triangle is the first and most simple of right-lined Figures, and is so called by reason it has three Angles: And when we say simply a Triangle, without specifying of what Sort, we always mean a right-lined Triangle, which

Plate 1. which is compos'd of three Right-Lines; a curvilined Triangle being a Plane-Figure terminated by three Curve-Lines. *Euclid* treats only here of the right-lined Triangle, whereof he makes six Species, *viz.* three that are diversified by their Angles, and three by their Sides; as shall be shewn after we have explain'd other more compos'd Figures.

XXII.

Fig. 13 *A* Figure that has four Sides, *which is also called a* Quadrilateral Figure, *and a* Quadrangle, *is a Plane-Figure terminated by four Right-Lines; as ABCD.*

This Figure is called a *Quadrangle*, because, having four Sides, it has also four Angles. *Euclid* makes also several Species or Kinds of these, diversified by their Angles and Sides; which we shall explain after the Triangles.

XXIII.

Fig. 19 *A* Multilateral *(or many-sided)* Figure, *called also a* Polygon, *is a Plane-Figure, terminated by more than four Right-Lines; as ABCDEF.*

This Figure is called a *Polygon*, because, having several Sides, it has also several Angles; when it has five it is called a *Pentagon*; when it has six an *Hexagon*; and when seven an *Heptagon*; when eight an *Octagon*; when nine an *Enneagon* or *Nonagon*; and a *Decagon* when it has ten; when eleven an *Endecagon*; and a *Dodecagon* when twelve: And when such a Polygon has all its Angles and all its Sides equal, it is called *Regular*, and *Irregular* when there are any of them unequal.

XXIV.

Fig. 10 *Among* Trilateral *(or three sided) Figures, that is called an* equilateral Triangle, *which has its three Sides equal; as DEF: whereof the three Sides DE, DF, EF, are equal.*

An equilateral Triangle is the most simple of all right-lined Figures, and only of one Kind; and it is with this Triangle that *Euclid* begins his Propositions (it being his first) that he may by means of this Problem resolve several others, altho' he might also have solv'd them by an *Isosceles* Triangle; but he was resolv'd to make use of the most simple.

XXV.

Fig. 9 *An* Isosceles Triangle *is that which has only two Legs equal; as ABC, whereof the two Legs or Sides AB, BC, are equal.*

It is evident, that among the different Sorts of Triangles, the *Isosceles* stands in the second Rank; at least with relation to its Sides. It may be either right-angled;

acute-angled (or an *Oxygon*); or obtuse-angled (or an *Amblygon*): Because the Angle C, contain'd by the two equal Sides AC, BC, may be either right, acute, or obtuse. It also follows, that every *equilateral Triangle* is an *Isosceles*, but not that every Isosceles is equilateral. Plate 1. Fig. 9.

XXVI.

A Scalene Triangle *is that whereof the three Sides are unequal; as GHI, the three Sides whereof, GH, GI, HI, are unequal.* Fig. 11

It is evident that a *Scalene Triangle* may be right-angled, because it may have one of its Angles right; and also obtuse-angled, because it may have one of its Angles obtuse; and acute-angled, because all its Angles may be acute, as in the precedent Triangle GHI.

XXVII.

Moreover, among three-sided Figures, that is called a right-angled Triangle which has one Right-Angle: as MKL, wherein the Angle K is a Right-one. Fig. 12

It is evident, that a right-angled Triangle may be an *Isosceles*, because the two Sides KL, KM, which contain the Right-Angle K, may be equal: It may also be *Scalene*, because the same two Sides KL, KM may be unequal, as they really are in this Figure, which makes all the three Sides unequal, because the Hypothenuse LM is greater than either of the two other Sides, KL, KM, as we shall demonstrate in the 19th *Prop.* But it can't be *Equilateral*, because its three Angles would then be equal by the 5th *Prop.* and consequently each would be one third of two Right-Angles, and therefore acute; because all the three Angles of a Triangle taken together, are exactly equal to two Right-ones, by the 32d *Prop.*

XXVIII.

An Amblygon Triangle *is that which has one Obtuse-Angle; as ABC, wherein the Angle C is obtuse, or greater than a Right-Angle.* Fig. 9

Hence we may see also, as before, that an Ambligon Triangle cannot be Equilateral, but that it may be either Isosceles or Scalene. We may also learn that it cannot be right-angled, because one of its Angles are suppos'd to be obtuse, that is, greater than a Right-one: whence it necessarily follows, that the other two must be acute.

XXIX.

An Oxygon Triangle *is that which has all its Angles acute*

Plate 1. Fig. 10. *acute; as DEF, where each of its three Angles D, E, F, is acute.*

We may easily perceive by what has been said of a right-angled Triangle, that an equilateral Triangle must needs be an *Oxygon*, and that an *Oxygon* may be either *Isosceles* or *Scalene*. These two last Sorts of Triangles, *viz.* the obtuse-angled and the acute-angled (which have no Right-Angle) are commonly called *Oblique-angled Triangles*.

XXX.

Fig. 13 *Among* Quadrilateral *(or four-sided)* Figures, *that is called a* Square, *which has four Right-Angles, and the four Sides equal; as ABCD.*

A Square is the most simple, and at the same time the most capacious of all four-sided Figures: And as there can be but one Sort of Square, it is commonly made use of in *Practical Mensuration*, viz. in measuring Surfaces, to express their *Contents* or *Area's*, that is to say, *what they contain in Square Measure*, as in square Feet, Yards, Poles, *&c.* A Right-Line drawn from any Angle of a Square, to the opposite one, as AC, or BD, is called the *Diagonal* or *Diameter* of that Square; and the Point where two such Diagonals intersect, and cut each other into two equal parts at Right-Angles, is called its Centre. We understand by a square Foot, or one Foot square, a Square whereof each Side is one Foot long; as likewise by a square Pole, a Square whereof each Side is a Pole in length.

XXXI.

Fig. 15 *An* Oblong, *which is also simply called a* Rectangle, *is a Figure of four Sides, which has all the Angles right, but which has not all the Sides equal; as KLMN.*

These two Figures, *viz.* the *Square* and the *Oblong*, are called rectangular or right-angled, because they have all their Angles right; and they differ only in this, *viz.* that the Oblong has only its two opposite Sides equal; as KL and MN, likewise KN, LM; whereas the Square has all its Sides equal. They are of great use in the common Affairs of Life, as in Surveying and Carpentry, *&c.* we reduce Figures into Squares or Rectangles, in order to measure them: In Architecture, *&c.* we commonly make Chambers, Courts, Gardens, and Allies, in Form of Rectangles: And in other Arts we see Tables, Cabinets, Looking-Glasses, *&c.* in that Shape.

XXXII.

A Rhombus *is a Figure consisting of four equal Sides, whereof the Angles are oblique; as EFGH.* Plate I. Fig. 14.

This Figure in Heraldry is called a *Losange*, and differs from a Square in this, that its Angles are not right ones, as having two acute, *viz.* the two opposite ones E, G; and the two other opposite ones F, H, obtuse: And in this also, that there may be several Sorts of them, because their Angles may vary, or be greater or less *ad infinitum.*

XXXIII.

A Rhomboid *is a Figure of four Sides, whereof the two opposite ones are equal, without being either equilateral or rectangular; as ABCD, wherein the two opposite Sides AB, CD, are equal; as also the other two opposite ones AD, BC, and wherein the Angles are oblique.* Fig. 17.

It is evident that this Figure, as well as the precedent, has two Angles opposite to one another acute, *viz.* A and C; and the two other opposite Angles B, D, obtuse: And that it may likewise vary or be diversified an infinite Number of Ways.

XXXIV.

All other quadrilateral Figures, which have not the Properties of the precedent ones, are called Trapezia; *as EFGH.* Fig. 18.

The four precedent Figures, *viz.* the *Square*, the *Oblong*, the *Rhomb*, and the *Rhomboid*, which may all be called *Parallelograms*, because their opposite Sides are parallel, as shall be demonstrated in the 34th *Prop.* are commonly reckoned among regular Figures; and all the rest, which *Euclid* calls *Trapezia*, are irregular Figures; which we shall distinguish into two Sorts, calling that only a *Trapezium*, none of whose Sides are parallel to one another, and that a *Trapezoid*, which has two parallel Sides, as Fig. 20. ABCD, where AB, CD, are parallel.

XXXV.

Parallel Right-Lines *are those that being produc'd indefinitely on the same Plane, will never meet; as ABCD.* Fig. 16.

To make this Definition yet clearer, we may add, that two Right-Lines that are parallel to one another, do not only not meet any where on the same Plane, how far soever produc'd, but also that they are always (or every where) equidistant from one another. And as the Distance

ſtance of any two Lines is eſtimated by the ſhorteſt Line that can be drawn betwixt them, which will be a perpendicular one; it follows that all the perpendicular Lines drawn between two Parallels are equal.

POSTULATES.

EUCLID in this Book, as likewiſe in all the reſt, makes uſe only of a Right-Line and a Circle; the deſcription whereof is ſo eaſy, that he takes it for granted by way of Poſtulate, that any one may;

I.

'From a given Point draw a Right-Line to any other 'Point given.

II.

'That one may produce a given finite Right-Line 'indefinitely.

III.

'That one may deſcribe a Circle from any given *Centre*, and with any given *Radius*.

To theſe there are commonly added two Poſtulates more; but as they don't agree with the Definition we have given of a *Poſtulate*, which is, that it is the Principle of a *Problem*, as an *Axiom* is of a *Theorem*; we ſhall, with other Commentators of *Euclid*, place them among the number of

AXIOMS.

I.

Plate 1. Fig. 19.

THoſe *Magnitudes which are equal to any common one, are equal amongſt themſelves*; e. g. The two Lines AF, BC, are each equal to the ſame third Line AB, and therefore they are alſo equal to one another.

This Axiom may be made more general thus; Thoſe Magnitudes which are equal to the ſame common one, or to any Number of equal ones, are equal among themſelves.

Clavius adds to this Axiom theſe two others; *viz.* Any Magnitude that is leſs or greater than either of two equal ones, is alſo leſs or greater than the other; and reciprocally, if of two equal Magnitudes the one is greater or leſs than a third Magnitude, the other ſhall alſo be greater or leſs than that third.

To

To these two Axioms may be added the three following, which *Euclid* makes use of in several of his Demonstrations, *viz.*

A Magnitude is equal to another Magnitude, when it is neither greater nor less than that Magnitude.

2. A Quantity is greater than another Quantity, when it is neither equal nor less.

3. A Quantity is less than another Quantity, when it is neither equal nor greater.

II.

If to equal Magnitudes you add equal Magnitudes, the whole will be equal: As, if to two Lines, each whereof is five Foot long, you add two others, each of three Foot long, you'll have two equal Lines, each of eight Foot long.

III.

If from equal Magnitudes you substract or take away equal Magnitudes, the Remainders will be equal: e. g. As if from two Lines each of eight Foot long, you substract or cut off two Lines each of three Foot long, there will remain two Lines each of five Foot long.

IV.

If to unequal Magnitudes you add equal ones, the Wholes or the Sums shall be unequal: e. g. If to a Line of three Foot long, and to a Line of two Foot long, you add two Lines of four Foot, one to each, you'll have two Lines, one of seven Foot, the other of six Foot, which are unequal.

To this Axiom *Clavius* adds this other, *viz.* If to unequal Magnitudes you add unequal Magnitudes, *viz.* the greater to the greater, and the less to the less, the Wholes shall be unequal: As, If to a Line of five Foot you add a Line of four Foot, and to a Line of two Foot you add a Line of three Foot; you'll have for the first a Line of 9 Foot, and for the second a Line of five Foot, which are unequal.

V.

If from unequal Magnitudes you substract unequal Magnitudes, the Remainder shall be unequal: As, if from a Line of eight Foot, and from another of six Foot, you substract two Lines of two Foot each; there will remain two Lines,

one of six Foot, and the other of four Foot, which are unequal.

Clavius adds likewise to this Axiom the following; If from unequal Magnitudes you substract unequal Magnitudes, *viz.* the less from the greatest, and the greatest from the less, the Remainders shall be unequal; the first Remainder being greater than the second: As, If from a Line of eight Foot you substract a Line of two Foot, and from a Line of six Foot you substract a Line of four Foot; you'll have on one hand a Line of six Foot, and on the other a Line of two Foot; which is less than the first remaining Line of six Foot.

VI.

Magnitudes that are double, each of the same Magnitude, are equal among themselves.

Because equal Magnitudes may be each taken for the other, or for one and the same Magnitude. This Axiom may be more generally express'd thus; 'Magnitudes 'which are double, each of the same Magnitude, or of 'equal Magnitudes, are equal among themselves': Or yet more generally thus; 'Magnitudes which are double, 'triple, quadruple, *&c.* of the same or equal Magnitudes, 'are equal among themselves.' Reciprocally it is evident, that if of two equal Magnitudes the one is double, triple, or quadruple, *&c.* of a third Magnitude, the other shall be also double, triple, or quadruple of the same Magnitude.

VII.

Magnitudes which are each one half of the same Magnitude, are equal among themselves.

This Axiom may also be made more general; and we may say, That Magnitudes which are the Half, or one third Part, or a Quarter, *&c.* of the same Magnitude, or of equal Magnitudes, are equal among themselves. And reciprocally, equal Magnitudes are each one Half, one Third, or one Quarter, of the same Magnitude, or of equal Magnitudes.

VIII.

Magnitudes which every way agree, are equal.

The Sense of this Axiom is (for Example) That if two Lines being plac'd one upon the other, do so agree, as that all the Parts of the one correspond exactly to all the Parts of the other, so that neither surpasses (or is greater or less than) the other, those two Lines are equal. We may

may say the same of two Angles, of two Surfaces, or of two Solids, when one being plac'd upon the other, or supposed to penetrate the other, neither of them surpasses the other.

IX.

The whole is greater than any one of its Parts.

To this Axiom may be added this other, *viz.* 'That 'all the Parts taken together are equal to the whole:' that is to say, that the whole is equal to all its Parts taken together.

X.

All Right-Angles are equal to one another.

This is a Corollary of the Definition of a *Perpendicular*, which supposes, that it makes on the Line on which it falls two equal Angles, which we call right ones. Whence it follows, that a right-lined Angle, or a curvilined, or a mixt Angle, may be said to be a right one, when it is equal to a right one.

XI.

If one Right-Line cut two other Right-Lines, so that it makes with them (on the same Side) the two interior Angles (taken together) less than two Right-Angles; those two Lines being produc'd on that Side, shall at length meet each other.

That is to say, if the two Right-Lines AB, CD, are cut by a third Right-Line DE, so that the two interior Angles, *e.g.* those towards the Extremities B and D, *viz.* BFG, DGF, are (taken together) less than two right ones; the Lines AB, CD being produc'd towards the said Extremities B and D, will meet. Plate 1. Fig. 21.

As this Theorem is not self-evident, we shall not make use of it as a Principle, but shall demonstrate it in the 14 *Prop.* after the same way as we have done it already in *Dechales*, because that Demonstration seems to me very natural. Because therefore this Axiom of *Euclid* is not to take place here, we will substitute the following in its room.

XII.

All the Perpendiculars that can be drawn between two Parallels are equal.

This Axiom is to be understood of two Parallel Right-Lines, and of Right-Lines that are perpendicular to one of them: For it is evident from the Definition of Parallels, that if the two Right-Lines AB, CD are parallel, Fig. 26.

 and

and there be drawn to one of those two the Perpendiculars EF, GH, and as many others as you please, all those Perpendiculars shall be equal to one another.

XIII.

Two Right-Lines can't comprehend (or include) Space, or constitute a Figure.

It is evident, That two Right-Lines, meeting one another, can only make an Angle, which is not a Figure. We might add, That two Right-Lines can only meet in one Point; which is the chief Reason why they can't include Space, or form a Figure.

XIV.

If one Magnitude is double of another, and a Line added to the first, double of a Line added to the second, the one whole shall be double of the other. As if to a Line of six Foot, which is the double of a Line of three Foot, you add a Line of four Foot, which is double of a Line of two Foot, (to be added to the other) the whole ten Foot will be double of the other whole five Foot.

XV.

If one Magnitude be double of another, and a Part cut off from the first, double of a Part cut off from the second, the Remainder of the first shall be double of the Remainder of the second. As if from a Line of ten Foot, which is double of a Line of five Foot, you cut a Line of four Foot, which is double of a Line of two Foot, the Remainder six Foot shall be double of the Remainder three Foot.

We omit several other Axioms, because the precedent ones are sufficient for the Demonstrations we shall here have occasion to make use of, wherein these Axioms shall be cited at length. As for the Propositions, and the Books where they are to be found, we shall cite them only by two Numbers, the first whereof shall denote the Proposition, and the second the Book. As for Example, if we were to cite the third Proposition of the second Book, we shall only set down these two Numbers, *viz.* 3, 2. And after this Way Mathematicians have in all Parts of the Mathematicks cited the Propositions and Books of *Euclid*'s Elements. And when in any Book of the Elements, the Citation is made by one Figure only, it denotes the Number of the Proposition of the same Book that was cited before.

PRO-

PROPOSITIONS.

PROPOSITION I.

PROBLEM I.

TO *make an equilateral Triangle on any given finite Line.*

To make an equilateral Triangle, *e. g.* on the given Line AB; from one End of the Line, *viz.* A, describe an Arch of a Circle BCD, that shall pass thro' the other end B, and likewise from the end B describe the Arch of a Circle ACE, which shall cut the precedent Arch BCD in the Point C, from which draw to the two ends A and B, the Right-Lines AC, BC; and the Triangle ABC will be an equilateral one; that is, the three Sides AB, AC, BC will be equal. Plate 2. Fig. 22.

DEMONSRATION.

The Line AC is equal to the Line AB, *by the Definition of a Circle*; And also the Line BC is equal to the same AB. Therefore by *Ax.* 1. the two Lines AC, BC, and consequently AC, BC, AB are all three equal to one another: *Which was to be demonstrated.*

USE.

This Proposition may not only be of use to demonstrate the next, but also the 9th, 10th, and 11th. And it may also be of use in several other Cases, and those not inconsiderable ones: As for example, it may serve for dividing a Line into any given Number of equal Parts; which may be easily done thus, *e.g.*

To divide the given Line AB into five equal Parts, set off at pleasure on the indefinite Line CD five equal Parts from C to D, and upon the Line CD describe the equilateral Triangle CDE; and draw thro' the Points of Division of the Base CD, to the Angle C, as many Right-Lines, and you'll have an Instrument not only fit and proper Fig. 24.

Plate 2. Fig. 24. proper for quinqui-secting the Line AB, but also any other Line whatsoever that is less than the Base-Line CD after this way, *viz.* Cut off from the two Sides EC, ED, the two Lines EF, EG, each of them equal to the given Line AB, and draw the Right-Line FG, which will be equal to AB the Line propos'd, and will be quinquisected by the Lines drawn from the Angle E, thro' the Divisions of the Base CD.

The Demonstration of this Praxis depends upon the 3. 6. *Eucl.* But if any Reader is not yet acquainted with that Book, nor the way of cutting off a less Quantity from a greater, it will be sufficient to suppose the thing as done, to superimpose the lesser Line on the greater; for in Practice, we may, according to *Aristotle*, suppose what we know how to do, as already done. This Proposition may be made use of to measure an Horizontal Line on the Ground, which is only accessible at one End, as we shall shew in our *Practical Geometry*.

PROPOSITION II.

PROBLEM II.

To draw from a given Point, a Line equal to a Line given.

Fig. 23. TO draw from the given Point A, a Line equal to the given Line BC, draw the right Line AB, and by *Prop.* 1. describe upon the Line AB, the equilateral Triangle ABD. Describe from the Point B, thro' the Point C, the Arch of a Circle ICK, and produce the Side BD, to the Point E, in the Arch of the said Circle. Describe from the Point D, thro' the Point E, the Arch of the Circle GEFH, and produce the Side AD, to the Arch of the said Circle in F. I say the Line AF, is equal to the the given Line BC; and consequently the Problem is resolv'd.

DEMONSTRATION.

If from the two Lines DE, DF, which are equal, by the Definition of a Circle, you cut off the two Lines DA, DB, which are also equal by Construction, because they are Sides of the equilateral Triangle ABD, there will

will remain by *Axiom* 1. the two equal Lines AE, BE. Thus we know that the Line AF is equal to the Line BE; and as *by the Definition of a Circle*, the Line BC, is also equal to the same Line BE, it follows by *Axiom* 1. that the Line AF is equal to the Line BC. *Q. E. F. & D.* Plate 2.

USE.

This Proposition may serve as a Lemma for the following, and also to demonstrate the 5 and 20 Proposition, and on several other Occasions.

PROPOSITION III.

PROBLEM III.

Two unequal right Lines being given, to cut off from the Greater, a Part equal to the Less.

TO cut off from the given Line AB, a Part equal to the other given Line CD, which I suppose to be the least; draw by *Prop.* 2. from the Point A, the Line AE equal to CD, and describe from the said Point A, thro' the Point E, the Arch of a Circle GFH, which shall cut off from the greatest given Line AB, the Part AF equal to the lesser given Line CD. Fig. 25.

DEMONSTRATION.

The Line AF is equal to the Line AE, *by the Definition of a Circle*, and the Line CD is equal to the same Line AE, *by Construction*; therefore by *Ax.* 1. the Line AF is equal to the Line CD. *Q. E. F. & D.*

USE.

This Proposition will be of Use to demonstrate the 8, and its several other Cases, which are not worth the while to talk of here. We may say that this, as well as the precedent, may be made use of several Ways, which we shall here omit, because the Construction and Demonstration will always be the same.

Plate 2.

PROPOSITION IV.

THEOREM I.

If in two Triangles, two Sides of the one are equal to two Sides of the other, each to each, and the two Angles comprehended between those equal Sides are equal; the Base of the one shall also be equal to the Base of the other, and the other two Angles of the one, equal to the remaining two Angles of the other, each to each respectively, and the two Triangles shall be wholly equal to each other.

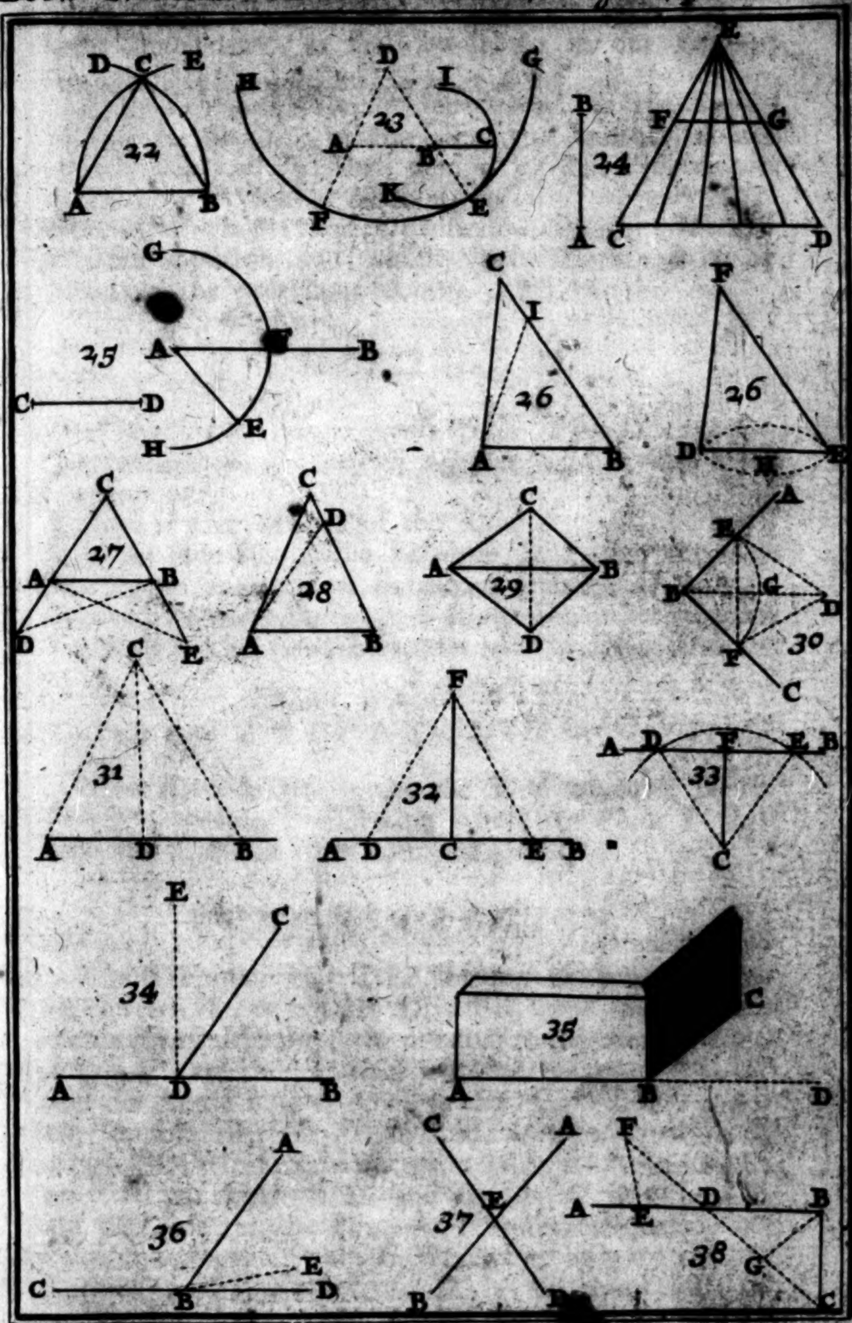
Fig. 26. I Say, that if the Side AC of the Triangle ABC, be equal to the Side DF of the Triangle DEF, and the Side BC equal to the Side EF, and the Angle C comprehended by those 2 Sides, equal to the Angle F; the Base AB shall be equal to the Base DE, and the Angle A to the Angle D, and the Angle B to the Angle E, and the whole Triangle ABC to the whole Triangle DEF.

DEMONSTRATION.

Imagine the Triangle ABC to be placed upon the Triangle DEF in such Manner that the Side AC shall just cover, or coincide with the Side DF, which may be done by *Ax.* 8. because those two Lines AC, DF are supposed equal; in which Case the Side CB shall fall exactly on the Side FE, because the two Angles C, F, are supposed equal; and the Point C falling upon the Point F, the Point B by *Ax.* 8. will fall upon the Point E, because the two Lines BC, EF are also suppos'd equal; for which Reason the Base AB will fall upon the Base DE, because if it fell either upon DGE, or DHE, two Lines would comprehend Space, contrary to *Ax.* 12. In like Manner by *Ax.* 8. the Base AB will be equal to the Base DE, and the Angle A to the Angle D, and the Angle B to the Angle E, and the whole Triangle ABC, to the whole Triangle DEF. *Q. E. D.*

USE.

This Proposition may be of use to demonstrate the following, and also the 8, 10, 14, 42. and several other Propositions of the following Books, but chiefly *Prop.* 6. of the 6th Book, which has a great Affinity with this. It may also serve to measure any inaccessible Line on the Ground, which you cannot go over by reason of some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some Impediment, as shall be shewn in our *Practical Geometry*.

As the Demonstrations which depend on the Supra-position (or placing) of one Line upon another, do not equally please all, we shall demonstrate the Propositions that follow in another Method, as also the very next *Theorem*, which *F. Tacquet* demonstrates by the Method of Supra-position, and which we shall demonstrate by Means of the precedent *Theorem*, as follows. Plate 2. Fig. 26.

THEOREM.

Two Triangles are always equal, if they have each one Side equal, and the two Angles, adjacent to that Side, equal, each to each.

I Say, if the Side AB of the Triangle ABC, be equal to the Side DE, of the Triangle DEF, and the adjacent Angle A equal to the adjacent Angle D, and the other adjacent Angle B equal also to the other adjacent Angle E; the two Triangles ABC, DEF shall be equal.

PREPARATION.

Upon the Side BC, make the Line BI equal to the Side EF, without considering where the Point I shall fall, and draw the right Line AI.

DEMONSTRATION.

The Triangles ABI, DEF, having the two Sides AB, BI equal to the two Sides DE, EF, and the Angle B comprehended between them, equal to the comprehended Angle E, are themselves equal *by the precedent Theorem*, and the Angle BAI is equal to the Angle EDF: and as we suppose that the Angle BAC is also equal to the Angle EDF, it follows by *Ax.* 1. that the Angle BAI is equal to the Angle BAC, and by *Ax.* 8. that the Line AI will fall on the Line AC, and consequently the Point I upon the Point C, whence it appears that BC is equal to BI: and because EF is also equal to BI, by constr. it follows by *Ax.* 1. that the two Sides BC, EF, are equal, and *by the precedent Theorem*, that the Triangle ABC is equal to the Triangle DEF. Q. E. D. See *Prop.* 26.

PROPO-

PROPOSITION V.

THEOREM II.

Plate 2. *In an Isosceles Triangle the two Angles above the Base are equal to one another, and the Sides being produc'd, the two Angles under the Base, are also equal to one another.*

Fig. 27. I Say that if the two Sides AC, BC of the Triangle ABC are equal to one another, and they be produc'd below the Base AB; the Angles ABC, CAB which are above the Base AB, will be equal to each other; and that the Angles ABE, BAD which are under the Base AB, will also be equal.

PREPARATION.

Set off upon the equal Sides AC, BC, prolong'd the two equal Lines AD, BE at pleasure, and draw the right Lines AE, BD.

DEMONSTRATION.

If to the equal Lines CA, CB, you add the two equal Lines AD, BE, it is Evident by *Ax*. 2. that the two Lines CD, CE will be equal, and by *Prop*. 4. that the two Triangles CDB, CEA will be also equal, because they have the Angle C common, and the two Sides CD, CB equal to the two Sides CE, CA. Wherefore the Base BD will be equal to the Base AE, the Angle D to the Angle E, and the Angle CAE to the Angle CBD, and by *Prop*. 4. The two Triangles ABD, BAE will be also equal, because they have the two Sides AD, BD equal to the two Sides BE, AE, and the contain'd Angle D equal to the contain'd Angle E. Wherefore the Angles DAB, ABE will be equal. *Which was one of the things to be demonstrated*. And the Angles ABD, BAE will also be equal, which being subtracted or taken away from the two Angles CBD, CAE, which were demonstrated to be equal, there will remain by *Ax*. 3. the two equal Angles CBA, CAB. *Which remain'd to be demonstrated.*

COROL.

Plate 2.

COROLLARY.

It follows from this Proposition, that an Equilateral Triangle, or one that has all its three Sides equal, is also Equiangular, or has all its three Angles also equal, because, as we have already observ'd elsewhere, every Equilateral Triangle is an Isosceles one.

USE.

An Isosceles Triangle may be made use of instead of an Equilateral one to divide a given Line, or a given Angle, into two equal Parts; as also to draw a Perpendicular to any Line given. The Use also of the Sector or Compasses of Proportion is founded on the Nature of an Isosceles Triangle: and thence likewise we calculated our Table of Plane Angles; the Use whereof we have shewn in taking the Measure of an Angle upon the Ground. This Proposition will also serve us to demonstrate the 18th, 20th, and 24th Propositions; and several others in the following Books.

PROPOSITION VI.

THEOREM III.

If a Triangle has two equal Angles, the Sides opposite to them will be also equal.

I Say if the two Angles ABC, BAC of the Triangle ABC, are equal to one another, the Sides BC, AC which subtend them, that is, which are opposite to them, shall also be equal to one another. Fig. 28.

PREPARATION.

On the Side BC set off the Line BD equal to the other Side AC, without considering where the Point D shall fall, and draw the right Line AD.

DEMONSTRATION.

The Triangles ABC, ABD, having the two Sides AB, BD equal to the two Sides AB, AC, and the contained Angle B equal to the contained Angle BAC, are equal

Plate 2. Fig. 28.

equal to one another, by *Prop.* 4. whence the Angle BAD is equal to the Angle B: and as we suppose the Angle BAC to be equal to the Angle B, it follows by *Ax.* 1. that the Angle BAD is equal to the Angle BAC, and consequently that the Line AD will fall on the Line AC, and the Point D upon the Point C, and consequently that the Side BC is equal to the Line BD, by *Ax.* 8. and as the Side AC, is also equal to the Line BD, by *Constr.* it necessarily follows from *Ax.* 1. that the two Sides AC, BC. must be equal to one another. *Q.E.D.*

COROLLARY.

It follows from this Proposition, that every Equiangular Triangle is also Equilateral, that is, that every Triangle, that has its three Angles equal, has also its three Sides equal.

USE.

This Proposition may be very conveniently made use of to measure a Line on the Ground that has one of its Ends only accessible, as shall be shewn in our *Practical Geometry*. It may also be made use of to measure the height of a Tower situated on an Horizontal Plane, by means of its Shadow, which will always be equal to the height of the Tower, when the Sun is 45 Degrees only above the Horizon, which may easily be found by a Quadrant, or an Astrolabe, *&c.* for then you have an Imaginary Right-angled Triangle, the Hypothenuse whereof is one of the Sun's beams, which terminates the Shadow, and in which each of the acute Angles consists of 45 Degrees, which makes the two Legs of the Triangle, *viz.* The Tower and its Shadow, equal.

As the 7th Prop. *only serves by way of Lemma to the 8th, which may be demonstrated alone without it; we shall omit it here, as being of no other Considerable Use in Geometry, our Design being only to treat of what may be useful.*

PROP.

PROPOSITION VIII.

THEOREM V.

If two Triangles have two Sides of the one, equal to two Sides of the other, each to each, and their Bases equal; those two Triangles are equal, and the Angles contained under the equal Sides are equal.

I Say, that if the Side AC of the Triangle ABC, be equal to the Side AD of the Triangle ABD, and the Side BC to the Side BD, and the Base AB be common to them both, which is the same thing as to have equal Bases; the two Triangles ABC, ABD, shall be every way equal. Plate 2. Fig. 29.

PREPARATION.

Draw the right Line CD, which will fall here within the two Triangles ABC, ABD, for it may also fall without, or concide with the two equal Sides: But the Demonstration of all these Cases will be easy to any one that throughly understands the Demonstration of the Case we have here before us.

DEMONSTRATION.

Since the two Sides AC, AD are equal, as also the two Sides BC, BD, *by Hypoth.* the Angle ACD will be equal to the Angle ADC, and the Angle BCD will be equal to the Angle ADC, by *Prop.* 5. and by *Ax.* 2. the whole Angle ACB will be equal to the whole Angle ADB. Wherefore by *Prop.* 4. the two Triangles ABC, ABD, will be wholly equal. *Q. E. D.*

USE.

This Proposition may serve as a Lemma to the following, as also to make an Angle, at any given Point of a Line, equal to an Angle given, as shall be shewn in *Prop.* 23. and it will be of particular use in *Prop.* 5. of the 6th Book, with which it has a very great Affinity.

PROPO-

PROPOSITION IX.

PROBLEM IV.

To divide an Angle into two equal Parts.

Plate 2. Fig. 30.

TO divide the Angle ABC into two equal Parts, that is to ſay into two equal Angles, deſcribe at Pleaſure from the Point B, the Arch of the Circle EFG, and draw the right Line EF, whereon make (by *Prop.* 1.) the equilateral Triangle DEF, in order to find the Point D, thro' which, and thro' the Point B of the given Angle ABC, draw the right Line BD; I ſay, that Line will divide the given Angle ABC into two equal Parts, or the Angle ABD will be equal to the Angle DBC.

DEMONSTRATION.

The Side BE of the Triangle BDE is equal to the Side BF of the Triangle BDF, (by the Definition of a Circle) and the Side DE is equal to the Side DF, becauſe they are the Sides of an equilateral Triangle, and moreover the Side BD is common to the two Triangles. Therefore by *Prop.* 8. thoſe two Triangles BED BFD are equal, and the Angle DBE is equal to the Angle DBF. *Q. E. D.* See *Prop.* 30. 3.

USE.

Prob. 7. Introd.

You may have ſeen in our Practical Geometry the uſe of this Problem, in dividing the Circumference of a Semicircle into twelve equal Parts of 15 Degrees each, and conſequently the whole Circumference into 24 equal Parts, for it is the ſame thing to divide an Arch as an Angle, it being certain that the Arch EF, which meaſures the Angle ABC, is alſo at the ſame Time divided into two equal Parts in the Point G, by the Line BD. It is alſo by Means of this Problem that we divide the Circumference of a Circle into 32 equal Parts, for the 32 Points of the Nautical Compaſs. This Problem is alſo very uſeful in Dyalling, when beſides the Hour-Lines, we have a Mind to ſet off the half Hours, and Quarters of Hours.

SCHO-

SCHOLIUM.

Euclid only shews us how to bisect an Angle, or divide it into two equal Parts, as for the Trisection, or dividing it into three equal Parts, or any other Number of odd Parts, it is Geometrically impossible, *viz.* By only making use of a Circle and right Line, as *Euclid* does. We shall repeat here what we have said on this Point, in our Notes on *F. Dechales's Euclid.*

By this Word Geometrically, *we are here only to understand the Circle and right Line,* Euclid's *Geometry extending it self no farther. But by the Geometry of Monsieur* Descartes, *we are taught that the Solution of a Problem is Geometrical, when it is resolv'd by the most simple and natural Way possible, altho' besides the Circle (or the Circumference of a Circle) we make use of some other Curve Line; as for Example, of some one of the Conick Sections for solid Problems, because a solid Problem is of such a Nature as to admit of no simpler Solution. Thus those for Example that would Trisect an Angle, only by a Circle and right Line, shew that they are not very conversant in Geometry, this Problem being by its Nature a solid one.*

PROPOSITION X.

PROBLEM V.

To divide a given Line into two equal Parts.

TO divide the given Line AB into two equal Parts, describe thereon the equilateral Triangle ABC, by *Prop.* 1. and by *Prop.* 9. divide the Angle C into two equal Parts by the right Line CD, which will also divide the proposed Line into two equal Parts in D; so that the two Parts AD, BD shall be equal to one another. Plate 2. Fig. 31.

DEMONSTRATION.

The Side AC of the Triangle ADC, is equal to the Side BC of the Triangle CDB, because they are the Sides of an equilateral Triangle; and the Side CD is common to them both, and the contained Angle ACD is equal to the contained Angle BCD by Construct, Therefore by

by *Prop.* 4. the two Triangles ADC, BDC are equal to one another, and the Base AD is equal to the Base BD. Thus the Line AB is divided into two equal Parts in D. *Q. E. D.*

USE.

This Problem may be very conveniently made use of, to draw thro' any Point assign'd without a given Line on the Ground, or on Paper, a Perpendicular, as may be seen in our Practical Geometry *on the Ground*, and as shall be shewn on Paper in *Prop.* 12. *Euclide* also makes use of it in his Preparation for the Demonstration of the 16 *Prop.* and it is used for several other Operations in Practice.

PROPOSITION XI.

PROBLEM VI.

From a given Point in a given Line to erect a Perpendicular.

Plate 2. Fig. 32.

TO draw a Perpendicular thro' the given Point C upon the given Line AB, set off at Pleasure on AB the two equal Lines CD, CE and by *Prop.* 1. Describe on the Line DE the Equilateral Triangle DEF, in order to find the Point F, thro' which, and the given Point C, draw the right Line CF, and that shall be the Perpendicular required, so that the two Angles DCF, ECF shall be equal to one another.

DEMONSTRATION.

The three Sides of the Triangle FCD, are equal to the three Sides of the Triangle FCE, the Side CE being equal to the Side CD by Construction, and the Side EF to the Side DF, because they are the Sides of an Equilateral Triangle, the Side CF being common. Therefore by *Prop.* 8. the two Triangles FCD, FCE are equal to one another, and the Angle DCF is equal to the Angle ECF. *Q. E. D.*

USE

USE.

The use of a Perpendicular is so common both in Mathematicks, and all Practical Arts, that he must have been but little conversant among Men, that does not know something of it. We make use of it in the 46 *Prop.* for drawing two Lines perpendicular to one another, in order to make a Square. And there is scarce any thing perform'd in Practical Geometry, without having occasion to draw a Perpendicular. We may say the same in Relation to *Fortification* and *Perspective*; and in *Dialling* we always begin by drawing two perpendicular Lines, if we are to make a Quadrant on any Plane by Geometrical Rules. Moreover *Stone-Cutters*, *Masons*, and several other Artificers have almost always their Squares in their Hands, to square their Works by.

PROPOSITION XII.

PROBLEM VII.

From a given Point, taken at Pleasure without a given right Line, to draw a Perpendicular to that Line.

TO draw from the given Point C, a Perpendicular to the given Line AB, describe at Pleasure from the Point C, the Arch of the Circle DE, which shall cut the given Line AB in two Points, as in D and E; and having by *Prop.* 10. divided the Line DE into two equal Parts in the Point F, draw from that Point, *viz.* F, to the given Point C, the right Line CF, I say that Line will be the Perpendicular sought; so that the two Angles CFD, CFE, shall be equal to each other, and consequently right ones. Plate 2. Fig. 33.

DEMONSTRATION.

If you draw the right Lines CD, CE, it is evident from the 8 *Prop.* that the two Triangles FCD, FCE are equal, because the three Sides of the one are equal to the three Sides of the other: for the Side CF is common, and the Side DF is equal to the Side EF *by Construction*, and the Side CD is equal to the Side CE by the Definition of a Circle. Whence it follows that the

the Angle CFD is equal to the Angle CFE. *Q. E. F & D.*

USE.

This Problem is useful on several Occasions, but chiefly in Surveying, where in order to know the Area of a Triangle upon the Ground, they are oblig'd to let fall from one of its Angles a Perpendicular to the opposite Side, to measure its Length by, and afterwards to multiply it by half the Side on which it falls, as we shall shew more particularly in our Practical Geometry.

PROPOSITION XIII.

THEOREM VI.

If one right Line fall upon another, it will either make with it two right Angles, or two Angles, which taken together, will be equal to two right ones.

Plate 2. Fig. 34.

I Say, that the Line CD, which cuts the Line AB in the Point D, makes with the said Line AB at the Point D, the two Angles ADC, BDC, which are either right Angles, or (taken together) equal to two right ones.

DEMONSTRATION.

It is evident *from the Definition of a Perpendicular*, that if the Line CD be perpendicular to the Line AB, the two Angles ADC, BDC, are right ones; but if it be not perpendicular to the Line AB, draw by *Prop.* 11. from the Point D, the Line DE which shall be perpendicular to it, in order to have the two right Angles ADE, BDE, to which the Sum of the two Angles ADC, BDC is equal; whence it follows that the two Angles ADC, BDC taken together, are equal to two right ones. *Q. E. D.*

COROLLARY I.

It follows from this Proposition, that if one of any two Angles made by a Line that falls on another right Line, be acute as BDC, the other ADC shall necessarily be obtuse: and if one of those two be right, the other shall be so too: And lastly, if one be known, the other will

will be so too, by subtracting the known one from two right ones, that is to say from 180 Degrees, because a right Angle consists of 90 Degrees, as being measured by one fourth Part of the Circumference of a Circle; which, as we have elsewhere shewn, consists of 360 Degrees.

COROLLARY 2.

It also follows, that if two right Lines intersect one another, they shall make four Angles, which taken together shall be equal to four right ones; for the two Angles on one Side are equal to two right ones, as we have already demonstrated, and by the same Reason, the two Angles on the other Side make also two right ones; and besides, all the four Angles are measured by the whole Circumference of a Circle, which measures (or contains) four right Angles. Whence it is easy to conclude, that all the Angles it is possible to form on a Plane by all the several right Lines that can terminate in the same Point, will altogether make four right Angles.

USE.

This Proposition may be of use not only for the following one, and several others, but also to measure an Angle on the Ground you cannot come within Side of: As for Example, the Angle ABC, made by the meeting of two Walls, for if you produce one of the two Sides or Walls AB, BC, by means of a Rope, or otherwise; for Example, AB towards D, and then measure the Angle CBD after the Method we have already shewn * elsewhere, the said Angle CBD being subtracted from 180 Degrees, the Remainder gives the Quantity of the Angle ABC, which was sought; as if *e. g.* the Angle CBD consists of 50 Degrees, by subtracting of 50 from 180, there will remain 130 Degrees for the Angle ABC, which was proposed to be found.

Plate 2. Fig. 35.

* Prol. 8. Introd.

PROPOSITION XIV.

THEOREM VII.

If at one Point of any right Line, two other right Lines meet, which make with it on both Sides two Angles equal together, to two right Angles; these two Lines being continued will make but one and the same right Line.

I Say, that if the two Lines BC, BD, meet at the Point B, of the Line AB, so that they make with that Line AB, the two Angles ABC, ABD, equal together to two right

Fig. 36.

Plate 2. Fig. 36. right Angles, these two Lines BC, BD, do meet at the Point B, directly, that is to say they make together one right Line.

PREPARATION.

Extend one of the two Lines BC, BD, as for Example, BC towards E, so that CBE be one right Line, without considering where the Line BE falleth.

DEMONSTRATION.

Since it is supposed that CBE is a right Line, the two Angles ABC, ABE, are together equal to two right Angles, *per Prop.* 13. and because the two Angles ABC, ABD, are together suppos'd also equal to two right Angles, it follows *per Ax.* 1. that the two Angles ABC, ABE, are together equal to the two ABC, ABD, taken together, and putting away the common Angle ABC, you will have *per Ax.* 3. the Angle ABE, equal to the Angle ABD, which shews *per Ax.* 8. that the Line BE, falls upon the Line BD, and that thus the two Lines BC, BD, are posited directly. *Which was the Thing to be prov'd.*

COROLLARY.

It follows from this Proposition, that if from one and the same Point of a right Line, two perpendicular Lines are drawn on both Sides, those two Perpendiculars will make a right Line.

USE.

This Proposition is the converse of the preceding, and may be useful in Practice, to know if three Points which are seen on the Ground, as B, C, D are in a right Line, when you cannot possibly pass to the two Extreams C,

Fig. 36. D, but only to the middle B; for then you need only chuse for the Sight a commodious Point upon the Ground, as A, and measure with a Graphometre or otherwise, the Quantity of the visual Angles, ABC, ABD, then

then add them together, and if their Sum is precisely 180 Degrees, it may be concluded that the three propos'd Points C, B, D, are in a right Line, otherwise they will be in the Circumference of a Circle, the Center whereof will be towards A, when that Sum shall be less than 180 Degrees, and contrariwise, when it shall be greater. Plate 2. Fig. 36.

PROPOSITION XV.

THEOREM VIII.

If two right Lines intersect, the opposite Angles at the Vertex will be equal to one another.

WHen two right Lines intersect, as AB, CD, which cut one another at the Point E, the two opposite Angles which they make at that Point E, as AEC, BED, are call'd *opposite Angles at the Vertex*, and are always equal. Fig. 37.

DEMONSTRATION.

The two Angles AEC, AED, are *per Ax.* 1. together equal to the two Angles, AED, BED, taken together, because each sum is equivalent to two Right-Angles, *per Prop.* 13. Wherefore by taking away the common Angle AED, there will remain *per Ax.* 3. the Angle AEC, equal to the Angle BED. *Which was to be shown.*

SCHOLIUM.

In the same manner may be shewn that the two other opposite Angles at the Vertex AED, BEC, are also equal to each other. But the Converse of this Proposition is likewise true, to wit, if at the same Point E, of the right Line AB, two other right Lines, EC, ED, meet together, which make with it the two opposite Angles at the Vertex AEC, BED, equal to each other, those two Lines EC, ED, will be in a right Line; because if to each of these two equal Angles AEC, BED, the common Angle AED, be added, it will be seen *per Ax.* 1. that the two AEC, AED are equal together to the two AED, BED, taken together, and because these two Angles AED, BED, make together two right Angles *per Prop.* 13. it follows that the two AEC, BED, are also together equal to two right Angles, and that *per Prop.* 14. the two Lines EC, ED, are in a right Line.

USE.

This Propoſition ſerves as a Lemma to the following, and ſerves likewiſe to meaſure an acceſſible Line upon the Ground, which cannot be perambulated by reaſon of ſome hindrance, as we ſhall ſhew in the *Practical Geometry*. It ſerves likewiſe to draw from a given Point without a given Line upon the Ground, a Perpendicular, as you ſhall ſee.

Plate 2. Fig. 38. To draw through the given Point C, a Line perpendicular to the given Line AB, draw through the Point C, to the Point D, taken at diſcretion upon the Line AB, the Line CD, and upon the ſame Line AB, the part DE, equal to the half CG, or DG, of the Line CD, continue the Line CD to F, ſo that the Line EF, may be equal to the Line DE, and make the Line DB, equal to the Line DF, to have the given Point B, through which, and through the given Point C, you are to draw the Line CB, which will be perpendicular to the propos'd Line AB; as will be found by drawing the right Line BG, which will be equal to the two GC, GD, by reaſon of the two equal and oppoſite Angles at the Vertex EDF, BDG, which renders the two Triangles EFD, DGB equal, *&c.*

Plate 3. Fig. 39. This Propoſition is likewiſe very uſeful to meaſure an inacceſſible Angle upon the Ground, as ABC. Thus, fix two Stakes in the Ground, in ſome commodious Place, as to the Points D, E, ſo that the three Points D, B, C, as well as the three A, B, E, be in a right Line, and meaſure with a Graphometre, or otherwiſe, the two Angles D, E, and ſubſtract their Sum from 180 Degrees, to have for a Remainder the third Angle DBE, or its equal and oppoſite at the Vertex ABC, which conſequently will be known.

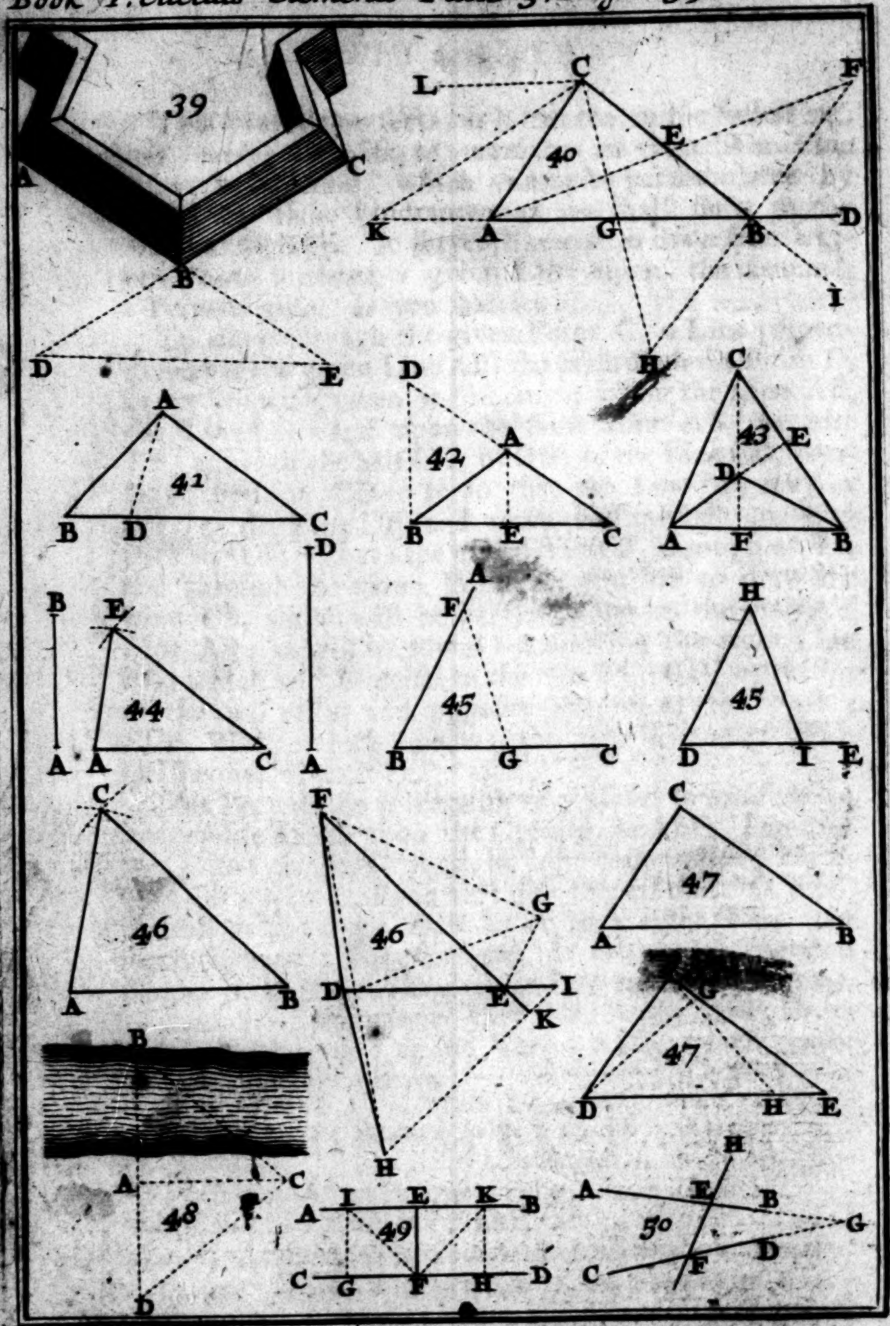
PROPOSITION XVI.

THEOREM IX.

One of the three Sides of a Triangle being produc'd, the exterior Angle is greater than either of the two interior oppoſite ones.

Plate 3. Fig. 40. I Say if you extend, for Example, the Side AB, of the Triangles ABC, towards D, the exterior Angle CBD, is greater than either of the two interior Oppoſite BAC, ACB.

PRE-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PREPARATION.

Having divided the Side CB, equally in two at the Point E, *per Prop.* 10. draw the right Line AE, and extend it to F, so that EF be equal to AE, and join the right Line BF. In like manner having divided the Side AB, equally in two at the Point G, draw the Line CG, and extend it to H, so that GH be equal to CG, and join the Line BH. Lastly, extend the Side BC, towards I. Plate 3. Fig. 40.

DEMONSTRATION.

Because the two Sides AE, CE, of the Triangle ACE, are equal to the two Sides EF, EB, of the Triangle EFB, *per constr.* and the included Angle AEC equal to the included Angle BEF, *per Prop.* 15. these two Triangles ACE, EFB, will be equal *per Prop.* 4. and the Angle ACE, will be equal to the Angle EBF, and consequently less than the Angle CBD. *Which was the Thing first to be demonstrated.*

In like manner, because the two Sides AG, CG, of the Triangle ACG, are equal to the two Sides BG, GH, of the Triangle BGH, *per constr.* and the included Angle AGC, equal to the included Angle BGH, *per Prop.* 15. these two Triangles BGH, ACG, will be equal *per Prop.* 4. and the Angle CAG will be equal to the Angle GBH, and consequently less than the Angle GBI. And because the Angle GBI, is equal to the Angle CBD, *per Prop.* 15. it follows that the Angle CAG, is likewise less than the Angles CBD. *Which remain'd to be prov'd.*

SCHOLIUM.

This Proposition and the following might be made appear more briefly, by considering them as Corollaries of the 32 *Prop.* which may be demonstrated independantly of these, as Father *Taquet* doth it.

It is evident that when the Interior Angle BCA, shall be the bigger, in which Case the Point A, will be farther off the Point B, this interior bigger Angle, to wit, BCK, will always be less than the exterior CBD, and that the Excess will not be so great; so that it will diminish continually, that is to say, that the Interior Angle will still more and more approach towards an Equality with the Exterior, in proportion as the Point A, becomes more remote from the Point B, till at length the Point A,

 being

Plate 3. Fig. 40. being infinitely remov'd from the Point B, in which Case the Line CA will be parallel to the Line AB; as for the purpose CL, the Angle BCL will be equal to the exterior CBD. From whence it evidently follows, that when the two Lines AB, CL, shall be parallel to each other, the two Angles BCL, CBD, which *Euclid* calls *Alternate Angles*, will be equal, and reciprocally that when these two alternate Angles BCL, CBD, shall be equal, the two Lines AB, CL, will be parallel.

USE.

Plate 2. Fig. 32. This Proposition serves not only to demonstrate the following and many others, but likewise to demonstrate, that from one and the same Point given, there cannot be drawn more than one Line perpendicular to a given right Line; because if from the Point F, cou'd be drawn, for Example, the two Lines FC, FE, perpendicular to the Line AB, the Exterior Angle FEB, which in this Case is a right one, would be equal *per Ax.* 10. to the interior opposite Angle C, which is also a right one, and yet it has been demonstrated to be greater.

It is likewise demonstrable by means of this Proposition, that from one and the same Point there cannot be drawn more than two equal Lines upon one Line given, because if from the Point F, cou'd be drawn for Example the three equal Lines, FD, FC, FE, each of the two Angles, FDC, FCE, wou'd be equal to the Angle FEC, *per Prop.* 5. Wherefore the Angle FCE, which is exterior with respect to the Triangle FCD, wou'd be equal to the interior opposite Angle FDC, and yet it hath been demonstrated to be greater. From whence it follows that a right Line and a Circumference of a Circle cannot intersect but in two Points.

PROPOSITION XVII.
THEOREM X.

In a Triangle any two Angles taken together are less than two right Angles.

Plate 3. Fig. 40. I Say, that the two Angles for Example ABC, BAC, of the Triangle ABC, are together less than two right Angles.

DEMONSTRATION.

For if the Side AB, is extended towards D, it will appear *per Prop.* 16. that the exterior Angle CBD, is greater than the interior opposite BAC. Wherefore if to each

each of these two unequal Angles CBD, BAC, the Angle ABC be added, you will have the two Angles BAC, ABC, less together than the two ABC, CBD, taken together, that is to say *per Prop.* 15. less than two right Angles. *Which was to be shewn.* Plate 3. Fig. 40.

COROLLARY.

It follows from this Proposition, that if in a Triangle one of the three Angles is a right one or even obtuse, each of the other two will of necessity be acute, and that in an Isoscele Triangle, each of the two equal Angles is also acute.

USE.

This Proposition begins to convince the Mind of the Truth of *Euclid's* 11 *Ax.* of which however we will give the Demonstration, when we shall have demonstrated the 34 *Prop.*

It serves also to prove that from one and the same Point, two Lines cannot be drawn perpendicular to one and the same Line, because if that were possible, you wou'd have a Triangle, where two Angles wou'd together be equal to two right ones, since each wou'd be a right one. Contrary to what we just now demonstrated.

It likewise serves to shew that if a Triangle hath an obtuse Angle, the Perpendicular drawn from one of the two acute Angles upon its opposite Side, will fall without the Triangle, towards the obtuse Angle, because otherwise you wou'd have a Triangle, where two Angles taken together wou'd be bigger than two right Angles, for the one wou'd be right, and the other obtuse: Contrary to what has been demonstrated.

PROPOSITION. XVIII.

THEOREM XI.

In any Triangle whatsoever, the greatest Side is opposite to the greatest Angle.

I Say, that if the Side BC, of the Triangle ABC, is for Example bigger than the Side AC, the Angle BAC, which respects the bigger Side BC, is bigger than the Angle B, which is opposite to the less AC. Fig. 41.

PRE-

PREPARATION.

Plate 3. Fig. 41. Cut off from the bigger Side BC, the Part CD, equal to the less AC, and join the right AD, which will necessarily be within the Triangle ABC.

DEMONSTRATION.

Because the two Sides CA, CD, of the Triangle ADC, are equal *per constr.* the two Angles DAC, ADC, will be also equal *per Prop.* 5. and because *per Prop.* 16. the exterior Angle ADC, is bigger than the interior opposite B, the Angle DAC, and much more the whole Angle BAC, will be bigger than the same Angle B. *Which was to be shewn.*

COROLLARY.

It follows from this Proposition, that in a Scalene Triangle, all the Angles are unequal. This also follows from the *6th Proposition*, because if there had been two equal Angles, there wou'd be likewise two equal Sides, and so the Triangle wou'd not be Scalene.

USE.

This Proposition serves not only for a Demonstration of the following which is its Inverse, but likewise very useful in Trigonometry, to be able to discern the greatest of the two Angles of a Triangle, without knowing it, which may be done, if the bigness, or only the Ratio of the opposite Sides be known, it being certain that the greatest of these two Angles will be that which shall be subtended by the greatest Side.

PROPOSITION XIX.

THEOREM XII.

In every Triangle the bigger Side is that which is oppos'd to the bigger Angle.

Fig. 41. I Say, that if the Angle BAC, of the Triangle ABC, is larger than the Angle B, the Side BC, opposite to the larger Angle BAC, is larger than the Side AC, opposite to the less Angle B.

DE-

DEMONSTRATION.

It is already evident that the Side BC, cannot be equal to the Side AC, because *per Prop.* 5. the Angle B wou'd be equal to the Angle BAC, which is suppos'd larger. It is also evident that the Side BC, cannot be less than the Side AC, because *per Prop.* 18. the Angle B, wou'd be larger than the Angle BAC, the which on the contrary is suppos'd larger. Since therefore the Side BC, cannot be equal nor less than the Side AC, it ought *per Ax.* 1. to be larger than the Side AC. *Which was to be prov'd.*

COROLLARY.

From this Proposition it follows, that of a right Angled Triangle, the greatest of the three Sides is the Hypotenuse, because the greatest of the three Angles is the Right Angle; and that in an Amblygone Triangle, the largest of all the Sides, is that which is opposite to the obtuse Angle, because this obtuse Angle is also the largest of the three Angles.

USE.

This Proposition serves as a Lemma to the following, and is very useful to demonstrate that the Perpendicular Line is the shortest of all those which can be drawn from one Point, to one and the same right Line; that is to say, that if the Line FC is perpendicular to the Line AB, it is less than the Line FE, which is oblique, because that Perpendicular FC, is opposite to the obtuse Angle FEC, which is less than the right Angle C, to which the oblique FE is opposite. Plate 2. Fig. 32.

PROPOSITION XX.

THEOREM XIII.

In all Triangles, any two Sides taken together, are greater than the third Side.

ALthough *Archimedes* hath taken this Proposition for an Axiom, we will however demonstrate it in *Euclid's* Manner. I say then the two Sides, for Example, AB, AC, of the Triangle ABC, taken together, are greater than the third Side BC. Plate 5. Fig. 42.

PRE-

PREPARATION.

Plate 5. Fig. 42. Lengthen one of the two Sides AB, AC, as AC, to D, so that the Line AD, be equal to the other Side AB, and join the right Line BD.

DEMONSTRATION.

Because the two Sides AB, AD, of the Triangle ABD, will be equal *per Constr.* the Angle D, is equal to the Angle ABD, *per Prop.* 5. and consequently less than the Angle DBC: Wherefore the Side CD, or the two AB, AC, are greater than the Side BC, *per Prop.* 19. *Which was to be shewn.*

SCHOLIUM.

Instead of extending the Side AC, you may *per Prop.* 9. divide the Angle BAC, equally in two by the right Line AE, and then you will find *per Prop.* 16. that the exterior Angle BEA, is larger than the interior opposite EAC, or EAB, and that consequently the Side AB is larger than the Side BE, *per Prop.* 19. You will find in the like Manner, that the Exterior Angle CEA, is bigger than the interior opposite EAB, or EAC, and that consequently the Side AC, is larger than the Side EC. From whence it is easy to conclude, that the two Sides AB, AC, are together larger than the two EB, EC, that is to say, than whole Side BC.

COROLLARY.

It follows from this Proposition, that a Right-Line is the shortest of all the Lines which can be drawn from one Point to another.

USE.

This Proposition serves as a Lemma to the following, whereof the preceding *Corollary* is likewise a Consequent, and I have not observ'd that it is of any considerable Use besides.

PROPOSITION XXI.

THEOREM XIV.

If from one Point taken at discretion within a Triangle, two Right-Lines are drawn to the Extremities of one of its Sides, they will be together less than the two other Sides of the Triangle, but they will make a larger Angle. Plate 3. Fig. 42.

I Say, that if from the Point D, taken at Pleasure in the Triangle ABC, the Right-Lines DA, DB, be drawn to the Extreams A, B, of the Side AB, their Sum DA+DB, will be less than the Sum AC+BC, of the two other Sides AC, BC; and that the Angle ADB, is bigger than the Angle ACB.

DEMONSTRATION.

In the Triangle AEC, which is had by extending AD, towards E, the Sum AC+CE is larger than AE, *per Prop.* 20. Wherefore if to each of these unequal quantities, you add EB, you will know *per Ax.* 4. that the Sum AC+BC, is larger than the Sum, AE+EB. Likewise in the Triangle DEB, the Sum DE+EB is larger than BD, *per Prop.* 20. and adding AD, you will have *per Ax.* 4. the the Sum AE+EB, larger than the Sum AD+BD. But the Sum AC+BC, has been demonstrated greater than the Sum AE+EB. Therefore the Sum AC+BC will with much more Reason be greater than the Sum, AD+BD. *Which was one of the two Things to be shewn.*

The exterior Angle ADB, is bigger than the interior opposite DEB, which being Exterior, with Respect to the Triangle AEC, is also bigger than the interior opposite ACE, *per Prop.* 16. Therefore with much more Reason, the Angle ADB, is bigger than the Angle ACB. *Which remain'd to be prov'd.*

SCHOLIUM.

If you draw the Right-Line CDF, it may be demonstrated in another manner, that the Angle ADB, is bigger than the Angle ACB: If you consider that the exterior Angle ADF, is bigger than the interior Opposite ACD, *per Prop.* 16. and that likewise the exterior Angle BDF, is bigger than the interior Opposite BCD, to con-

Plate 3. Fig. 43. conclude from thence, that the Sum of the two Angles, ADF, BDF, that is to say the whole Angle ADB, is bigger than the Sum of the two ACD, BCD, or than the whole Angle ACB.

If upon the same Base AB, another Triangle be describ'd within the Triangle ADB, and so on, it would be demonstrable as before, that the two Sides of the later Triangle, would be together less than the two Sides of the preceeding Triangle. From whence it is easy to conclude, that the Sum of the two Sides still continuing to diminish as far as the Right-Line AB, this Right-Line AB, is the least of all those which can be drawn through its two Extremities A, B.

USE.

This Proposition serves to demonstrate a Case of the 8. 3. *Prop.* it may serve also to demonstrate the 21. 11. *Prop.* and we shall make very good use of it in Spherical Trigonometry, to demonstrate that in a Spherical Triangle, the three Angles taken together are bigger than two Right-Angles.

PROPOSITION XXII.

PROBLEM VIII.

To describe a Triangle of three given Lines, whereof the bigger ought to be less than the Sum of the other two.

Fig. 44. TO describe a Triangle, whose three Sides shall be equal to the three Lines, AB, AC, AD, the biggest whereof AD, ought to be less than the Sum of the two others, AB, AC, otherwise the Problem wou'd be impossible, because *per Prob.* 20. in every Triangle, the Sum of any two Sides is greater than the third, if you would have the second given Line AC, serve for a Base to the Triangle that is search'd for, describe from its Extremity A, an Arch of a Circle at the opening of one of the two other given Lines AB, AD, as of AB; and with the Interval of the last given Line AD, describe from the other Extremity C, another Arch of a Circle, which shall intersect the first, at the Point E, from which you must draw to the two Points A, C, the Right-Lines EA, EC, and the Triangle ACE, will be that which is sought for.

DE-

DEMONSTRATION.

Since the Arch of the Circle describ'd from the Point A, was made with the Interval of AB, the Side AE, ought of Necessity to be equal to the Line AB; and in like Manner the Side CE, is equal to the Line AD: so the three Sides of the Triangle ACE, are equal to the three given Lines AB, AC, AD. *Which was to be done and Demonstrated.* Plate 3. Fig. 44.

USE.

This Problem seems to be put here by *Euclid* for no other Reason but to resolve the following; because its made no Use of afterwards. But it may be very serviceable to describe a Figure equal to another, which for that Purpose, when it hath more than three Sides, ought to be reduc'd into Triangles by several Diagonals, or Right-Lines drawn from one Angle to another, to make other Triangles apart in the same Order, which shou'd have all the Sides equal to all the Sides of the Triangles, which will be found in the propos'd Figure. This may be likewise perform'd, by making a like Figure, when the propos'd Figure shall be projected; that is to say, when you wou'd raise an accessible Plane on the Ground, to wit, by taking on every Side, as many little Parts measur'd by a Scale, as the Sides of the Triangle of the propos'd Plan shall have Feet or Yards; as you have seen in *Prob.* 16. *Introd.*

PROPOSITION XXIII.

PROBLEM IX.

To make at a given Point of a given Right-Line, an Angle equal to a given Angle.

TO make at the given Point D, of the given Line DE, an Angle equal to the given Angle ABC, draw thro' the two Points F, G, taken at Discretion upon the Lines AB, AC, the right FG, and make *per Prop.* 22. from the three Lines BF, BG, FG, the Triangle DHI, so that the two Sides DH, DI, which are round about the given Point D, be equal to the two Sides BF, BG, which make the propos'd Angle B; and the Angle D, will be equal to the given Angle B. Fig. 45.

DE-

DEMONSTRATION.

Plate 3. Fig. 45.

Since the three Sides of the Triangle DHI, are equal *per Constr.* to the three Sides of the Triangle BFG; these two Triangles BFG, DHI, will be equal to one another, *per Prop.* 8. and the Angle D, will be equal to the Angle B, because they are opposite to the equal Sides. *Which was to be done and demonstrated.*

USE.

This Proposition serves not only for the Demonstration of the following, and to resolve the 42, but likewise for the Determination of *Prop.* 33, and 34. *l.* 3. and also *Prop.* 2. and 3. *l.* 4. It serves likewise to raise an accessible Plan, or inaccessible which is on the Ground, as you have seen in *Prob.* 16, 17. *Introd.*

Lastly, It serves in *Dialling*, in *Perspective*, in *Fortification*, and in all the other Parts of the Mathematicks, where the Rule and Compasses are us'd, and principally in *Geodæsia*, that is to say, in Surveying of Lands, the Operations thereof for the most Part wou'd be impossible, if you cou'd not make one Angle equal to another, or of such a Number of Degrees as you wou'd.

PROPOSITION XXIV.

THEOREM XV.

If two Triangles have two Sides equal to two Sides, each to each, that which hath the greatest Angle contain'd by those two equal Sides, has the greatest Base.

ALthough this Proposition be as a Corollary of the fourth, nevertheless as that Corollary depends properly upon nothing but the Senses, and that its Certainty ought to be evident to Reason, and the Principles whereon it dependeth, we shall demonstrate it in *Euclid*'s Manner, thus,

Fig. 46.

I say then, that if the Side AC, of the Triangle ABC, be equal to the Side DF, of the Triangle DEF, and the Side BC, equal to the Side EF; but that the included Angle ACB, be greater than the included Angle DFE; the Base AB, will be greater than the Base DE.

PRE-

PREPARATION.

Make *per Prop.* 23. at the Point F, of the Line DF, the Angle DFG equal to the Angle C, with the Line FG, which will necessarily fall without the Triangle DEF, because the Angle DFE is suppos'd less than the Angle C. Make the Line FG equal to the Line BC, and join the right Line DG. Plate 4. Fig. 46.

DEMONSTRATION.

Because the Line DF is equal to the Line AC, *per. Sup.* and the Line BC equal to the Line FG, *per constr.* and likewise the Angle C, equal to the Angle DFG, *per constr.* the two Triangles ABC, DEF, will be equal to one another, *per Prop.* 4. and the Base AB, will be equal to the Base DG.

Because the Sides EF, FG, are equal each to the same Side BC, *per constr.* it follows *per Ax.* 1. that the Sides FG, FE, are equal, and that *per Prop.* 5. the Angle FEG, is equal to the Angle FGE, and consequently greater than the Angle DGE, which with much more Reason will be less than the Angle DEG, therefore by *Prop.* 19. the Line DG, or AB, its equal, as hath been demonstrated, is greater than DE. *Which was to be shewn.*

USE.

This Proposition serves not only to demonstrate the following, which is its Inverse, but likewise to demonstrate a Case of *Prop.* 7. and 8. *l.* 3. and a Case of *Prop.* 15. *l.* 3.

PROPOSITION XXV.

THEOREM XVI.

Of two Triangles which have two equal Sides, each to each, that which hath the greater Base, hath the Angle opposite to that Base, also greater than the Angle opposite to the lesser Base.

I Say, that if the Side AC of the Triangle ABC, be equal to the Side DF of the Triangle DEF, and the Side BC equal to the Side EF; but the Base AB greater than the Base DE; the Angle C is greater than the Angle DFE.

DEMONSTRATION.

Plate 3. Fig. 46.

First, The Angle C cannot be equal to the Angle DFE, because by *Prop.* 4. the Base AB wou'd be equal to the Base DE, and yet it is suppos'd to be greater. Nor can the same Angle C be less than the Angle DFE, because by *Prop.* 24. the Base AB wou'd be less than the Base DE, and yet it is suppos'd to be greater. Therefore by *Ax.* 1. the Angle C is greater than the Angle DFE. *Which was to be demonstrated.*

SCHOLIUM.

Altho' this Demonstration be not direct, it doth not fail to convince the mind of the truth of this Proposition, and it seems that *Euclid* puts it here only for its Easiness.

If you wou'd have a direct one, make at the Point D, *per Prop.* 23. the Angle EDH equal to the Angle A, by the Line DH, equal to the Line AC, or DF its equal *per Sup.* and having extended the Base DE to I, so that the Line DI, be equal to the Base AB, join the right-Line HI, which is here cut at K, by the Side EC extended, join likewise the right Line FH.

This Preparation being made, it will appear that since the two Sides DH, DI, of the Triangle DHI, are equal to the two Sides AC, AB, of the Triangle ABC, and the compriz'd Angle HDI, equal to the compriz'd Angle A, *per. constr.* these two Triangles ABC, HDI, are equal to one another, *per Prop.* 4. and consequently the Side BC, or EF equal to the Side HI, and the Angle C equal to the Angle DHI. From whence it follows that the Line KF is greater than the Line KH, and that *per Prop.* 18. the Angle FHK is greater than the Angle HFK; and because that *per Prop.* 5. the Angle DFH is equal to the Angle DHF, by reason of the two equal Sides DF, DH, *per constr.* it follows *per Ax.* 4. that the whole Angle DHK, or the Angle C, which hath been demonstrated equal to it, is greater than the whole Angle DFE. *Which was to be demonstrated.*

PRO-

Plate 3.

PROPOSITION XXVI.

THEOREM XVII.

The Triangle which hath two Angles, equal to those of another, and one Side, similarly posited, likewise equal, is equal to it every Way.

I Say, that if the Angle A of the Triangle ABC, be equal to the Angle FDE of the Triangle DFE, and the Angle B equal to the Angle E, and likewise the Side AB equal to the Side DE, which are compris'd between the two equal Angles, or the Side AC equal to the Side DF, which are opposite to the two equal Angles B, E, these two Triangles ABC, DEF, are intirely equal. Fig. 47.

PREPARATION.

Upon Supposition that the Side AB is equal to the Side DE, take on the Side EF, the Line EG, equal to the Side BC, without considering where the Point G falleth; and join the Line DG; and upon Supposition that the Side AC is equal to the Side DF, take on the Side DE, the Line DH, equal to the Side AB, without considering where the Point H falleth, and join the Line FH.

DEMONSTRATION.

Because *per Sup.* 1. the Side AB of the Triangle ABC, is equal to the Side DE of the Triangle DEF, and the Angle B, equal to the Angle E, and that the Side EG, hath been made equal to the Side BC, the two Triangles ABC, DGE, will be equal to one another, *per Prop.* 4. and the Angle GDE will be equal to the Angle A, and consequently to the Angle FDE. From whence it follows that the Line DG, falleth on the Line DF, and consequently the Point G upon the Point F. Wherefore the Side EF will be equal to the Side EG, and consequently to the Side BC, and *per Prop.* 4. the Triangle ABC will be equal to the Triangle DEF. *Which is one of the Cases which was to be demonstrated.*

Because *per Sup.* 2. the Side AC of the Triangle ABC, is equal to the Side DF of the Triangle DFH, and the comprehended Angle A equal to the comprehended Angle FDE, and that the Side DH has been made equal to the Side AB, these two Triangles ABC, DFH, will be

equal to one another, *per Prop.* 4. and the Angle DHF, will be equal to the Angle B, and consequently to the Angle E, which is suppos'd equal to the Angle B. From whence it follows that the Point H, ought to fall upon the Point E, otherwise an exterior Angle wou'd be had equal to its interior opposite, which is contrary to *Prop.* 16. and that consequently the Side DH, or AB, is equal to the Side DE. Wherefore *per Prop.* 4. the Triangle ABC is equal to the Triangle DEF. *Which remain'd to be prov'd.*

Plate 3. Fig. 47.

USE.

Plate 2. Fig. 31.

Euclid doth not often make use of this Proposition, tho' it be very useful upon many occasions. It may serve to demonstrate that in an *Isosceles* Triangle, as ABC, if the Angle C, included by the two equal Sides AC, BC, be divided equally in two by the right Line CD, this right Line CD, will cut the Base AB at right Angles, and equally in two at the Point D; or if from the same Angle C, you draw upon the Base AB, the Perpendicular CD; this Perpendicular CD, will divide the Base AB equally in two, by reason of the two equal Triangles ADC, BDC, which have the Angles equal, each to each, and an equal Side similarly posited, to wit, the common Side CD.

We shall make use of this Proposition also in Dialling, to demonstrate the manner, which we shall there shew, to find the dividing Center of a Right-Line, which represents upon a Plane a great Circle of the Sphere; and the same Proposition may be very useful to measure on the Ground, a Line which is only accessible at one of its two Extreams as AB, which I suppose to be accessible towards A, where you are to make, by means of a Graphometre, or otherwise, the Right-Angle BAC, with the Line AC, of a discretionary Length; after which you ought to remove your self to the Point C, to measure the Quantity of the Angle ACB, and to make one equal to it on the other Side at the same Point C, as ACD, with the Line CD, which being extended as much as there shall be occasion for, it will meet the Line AB, also extended, in some Point as D; and then there will be nothing more to be done but to measure with a Cord, or otherwise, the Line AD, which will be equal to the propos'd Line AB, by reason of the Equality of the two Triangles ACB, ACD, which have equal Angles, and one equal Side similarly posited, to wit, the common Side AC.

Plate 3. Fig. 48.

PRO-

Plate 3.

PROPOSITION XXVII.

THEOREM XVIII.

If one Right-Line falling upon two other Right-Lines, make the interior alternately opposite Angles equal to each other: these two Lines will be parallel to each other.

I Say, that if the Right-Line HF, cut the two AB, CD, so that the two interior alternately opposite Angles AEF, EFD, which are call'd *Alternate Angles*, are equal to each other; these two Lines AB, CD, are parallel to each other. Fig. 50.

DEMONSTRATION.

For if the two Lines AB, CD, were not parallel, they wou'd, being extended, meet in some Point, as in G, and then they wou'd make the Triangle EFG, whereof the exterior Angle AEF wou'd be equal to its interior opposite EFG, contrary to what hath been demonstrated in *Prop.* 16. Thus the two Lines AB, CD, cannot meet together, and *per Def.* 35. they ought to be parallel to each other. *Which was to be demonstrated.*

SCHOLIUM.

This Proposition is a result of the remark that we have made in *Prop.* 16. It may be demonstrated directly, by drawing *per Prop.* 12. from the Point F, the Line FI, perpendicular to the Line AB. and by taking the Line FK, equal to the Line EI, and joining the Line EK; after which it will be known *per Prop* 4. that the two Triangles EIF, EKF, are equal to each other, by reason of the two Sides EI, EF, equal to the two KF, EF, and by reason of the compris'd Angle IEF, equal to the compris'd Angle EFK, *per Sup.* From whence it follows that the Angle K is equal to the Angle I, and consequently a right one, and that the Line EK is perpendicular to the Line CD, and moreover that this perpendicular EK, is equal to the Line FI, which is also perpendicular to the Line AB, *per Constr.* which makes that the two Lines AB, CD, are equally remote from one another, and consequently parallel. Plate 4. Fig. 51.

USE.

It may be known by this Proposition, when two Lines upon the Ground or upon Paper, are Parallels, which

will happen when the alternate Angles ſhall be equal. It ſerves alſo to draw thro' a given Point a Line parallel to a given Line, as you will ſee in *Prop.* 31. and as you have already ſeen in *Prob.* 3. *Introd.* It ſerves alſo to demonſtrate *Prop.* 32. and ſeveral others, as you ſhall ſee hereafter.

Plate 4. Fig. 41.

PROPOSITION XXVIII.

THEOREM XIX.

If one Right-Line cutting two other Right-Lines, make with them the exterior Angle equal to its oppoſite interior on the ſame Side, or the two Interiors on the ſame Side, equal together to two Right-Angles; theſe two Right-Lines will be parallel to one another.

Plate 4. Fig. 51.

I Say, that if the Right-Line GF, cut the two AB, CD, ſo that the exterior Angle GEB, be equal to the interior oppoſite of the ſame Side EFD, or that the two Interiors of the ſame Part BEF, EFD, be together equal to two right ones, the two Lines AB, CD, are parallel.

DEMONSTRATION.

Since the Angle EFD is equal to the Angle GEB, *per Sup.* and the Angle AEF equal to the ſame Angle GEB, *per Prop.* 15. it follows *per Ax.* 1. that the Angle AEF is equal to the Angle EFD, and *per Prop.* 27. that the Lines AB, CD, are parallel to each other. *Which is one of the two Things which was to be demonſtrated.*

Since the two Angles BEF, EFD, are alſo together equal to two right Angles, *per Sup.* and that the two BEF, AEF, are alſo together equal to two right ones, *per Prop.* 13. it follows *per Ax.* 3. that if from theſe two equal Sums you ſubſtract the common Angle BEF, there will remain the Angle AEF, equal to the Angle EFD, and *per Prop.* 27. the two Lines AB, CD, are parallel. *Which remain'd to be prov'd.*

USE.

This Propoſition hath the ſame Uſes as the precedent, and moreover it ſerves to convince the Mind of the truth of *Euclid*'s eleventh *Axiom*, for it is evident that the two interior Angles BEF, EFD, which are on one and

and the same Side being equal together to two right Angles, the Lines AB, CD, are Parallel; and that those two Angles cannot become so little less than two right ones, as that the two Lines AB, CD, will not meet (being extended) on the same Side. Plate 3. Fig. 50.

LEMMA.

The Right-Line which is perpendicular to one of two Parallels, is also perpendicular to the other.

I *Say, that if the Line EF, be perpendicular to one of the two Parallels AB, CD, as for Example to the Line CD, it is also Perpendicular to the Line AB.* Plate 3. Fig. 49.

PREPARATION.

Take upon the Line CD, the two equal Lines FG, FH, of a discretionary bigness, and draw thro' the two Points G, H. per Prop. 11. *the Lines GI, HK, perpendicular to the same Line GD. Join the right Lines FI, FK.*

DEMONSTRATION.

Because the Side FG, of the Triangle FGI, rightangled in G, is per construct. *equal to the Side FH of the Triangle FHK, rightangled in H, and the Side GI, equal to the Side HK,* per Ax. 11. *these two rightangled Triangles FGI, HFK, will be equal to one another,* per Prop. 4. *and the Base FI will be equal to the Base FK, and the two Angles GFI, FHK, will be equal, the which being subducted from the two Angles GFE, HFE, which are equal,* per Def. 10. *because they are right ones,* per Sup. *there will remain,* per Ax. 3. *the two equal Angles EFI, EFK, and* per Prop. 4. *the two Triangles IEF, KEF, will be equal to each other, because they have the common Side EF, the Side FI equal to the Side FK, and the compris'd Angle EFI equal to the compris'd Angle EFK, as hath been demonstrated. Wherefore the Angle IEF will be equal to the Angle KEF, and* per Def. 10. *these two Angles will be right ones, and the Line EF will be perpendicular to the Line AB.* Which was to be demonstrated.

Plate 4. Fig. 11

PROPOSITION XXIX.

THEOREM XX.

If one Right-Line intersect two Parallels, the alternate Angles will be equal to one another; the exterior Angle will be equal to the interior opposite on the same Side; and the two Interiors, on the same Side, will together be equal to two Right-Angles.

I Say, that if the Right-Line GF, cut the two Parallels AB, CD, the alternate Angles AEF, EFD, are equal to each other; the exterior Angle GEB is equal to the interior opposite on the same Side EFD; and that the two Interiors on the same Side BEF, EFD, are together equal to two Right-Angles.

PREPARATION.

Draw from the two Points E, F, the Right-Lines EK, FI, perpendicular to the two Lines AB, CD.

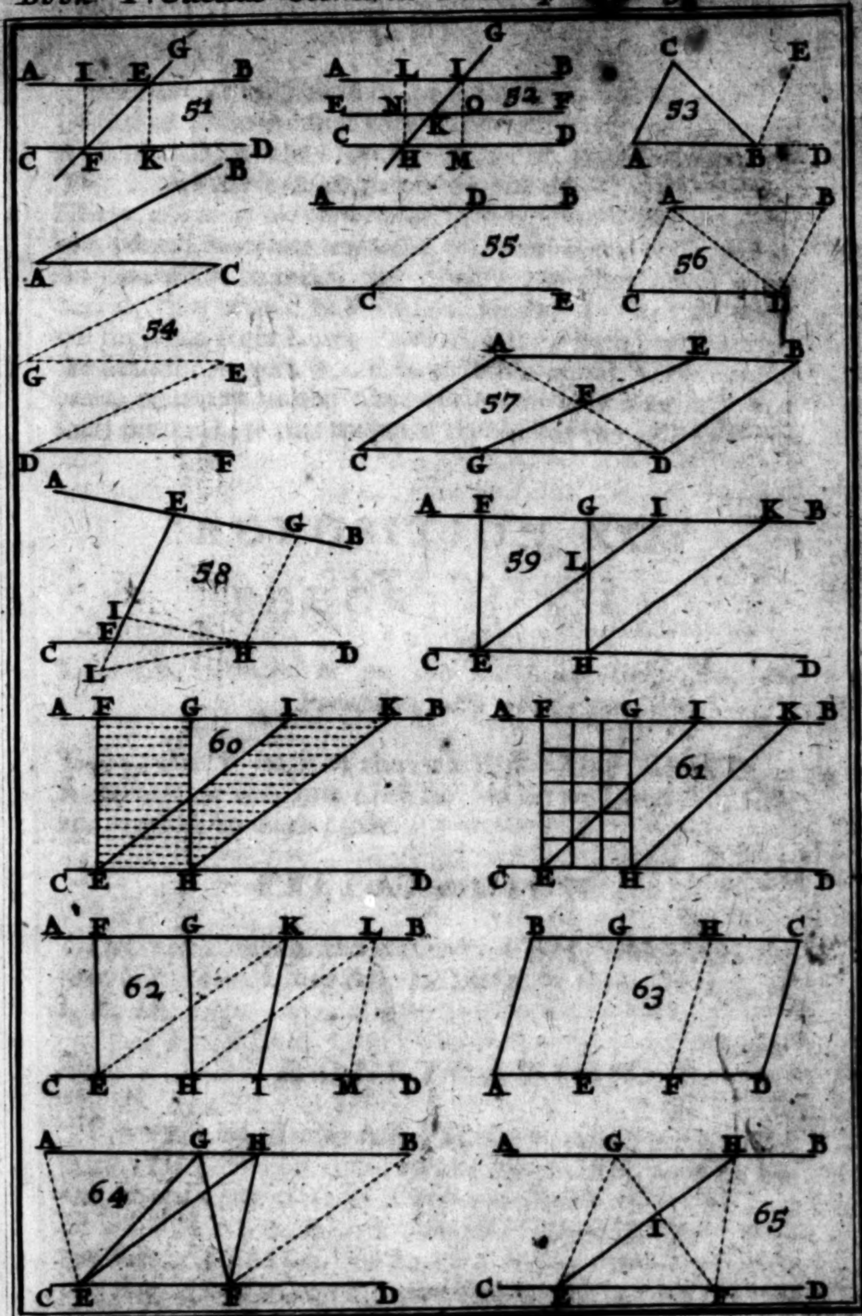
DEMONSTRATION.

The two Lines FI, KE, are equal to each other, *per Ax.* 11. and each will be, *per preceeding Lemma,* perpendicular to the two Parallels AB, CD; also the two Angles IFK, EKF, will be right ones, and consequently equal together to two right ones, wherefore *per Proposition* 28. the two Lines FI, KE, are Parallels, to which the two IE, FK, being perpendicular, are equal to each other, *per Ax.* 11. Wherefore *per Prop.* 8. the two Triangles FIE, FKE, will be equal to one another, and the Angle IEF will be equal to the Angle EFK. *Which is one of the three Things which was to be prov'd.*

Since the Angle AEF hath been demonstrated equal to the Angle EFD, and that it is also equal to the Angle GEB, *per Prop.* 15. it follows, *per Ax.* 1. that the Angle GEB is equal to the Angle EFD. *Which was likewise to be demonstrated.*

Lastly, Since the two Angles BEF, AEF, are together equal to two right ones, *per Prop.* 13. if instead of the Angle AEB, you take its alternate EFD, which has been demonstrated equal to it, it will appear that the two Angles BEF, EFD, are together equal to two right ones. *Which remain'd to be demonstrated.*

USE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USE.

We have already said in our Remarks upon the *Euclid* of Father *Dechales*, that this Proposition serves likewise to demonstrate the eleventh *Axiom* of *Euclid*, which is, That *if one Right-Line falling on two others, makes the two interior Angles of the same Side, less together than two right ones, these Lines being extended will meet on this Side*; for if they were not to meet, that is to say, if they never concurr'd, they wou'd be Parallels, *per Def.* 35. because they are suppos'd right Lines; and also as it hath been shewn, the interior Angles wou'd be together equal to two right ones, contrary to the Supposition of this Maxim. We shall better shew this towards the end of the 34 *Prop.* Plate 4. Fig. 51.

PROPOSITION XXX.

THEOREM XXI.

Right-Lines Parallel to one and the same Right-Line, are Parallel to each other.

I Say, that if each of the two Right-Lines AB, CD, is parallel to the same Line EF, these two Lines AB, CD, are parallel to each other. Fig. 52.

PREPARATION.

Draw at Pleasure the Right-Line GH, which cuts the propos'd three Lines AB, EF, CD, in three Points, as I, K, H.

DEMONSTRATION.

Since the two Lines AB, EF, are Parallel, *per Sup.* the Angle GIB will be equal to the Angle IKF, *per Prop.* 29. and since in like manner it is suppos'd that the two Lines EF, CD, are parallel, the Angle KHD, will be equal to the same Angle IKF. Whence it follows *per Ax.* 1. that the Angle GIB is equal to the Angle KHD, and that *per Prop.* 28. the two Lines AB, CD, are Parallels. *Which was to be demonstrated.*

SCHO-

SCHOLIUM.

Plate 4. Fig. 52.

This Propoſition may be demonſtrated otherwiſe, and very eaſily by drawing at pleaſure the two Lines LH, IM, perpendicular to the Line EF, which will alſo be perpendicular to each of the two Lines, AB, CD, *per preceding Lemma.*

The two Lines LN, IO, are equal to each other, *per Ax.* 11. as well as the two HN, MO: Wherefore *per Ax.* 2. the two Lines LH, IM, will be likewiſe equal to each other, and *per Def.* 35. the two Lines AB, CD, will be parallel to each other. *Which was to be demonſtrated.*

The three Lines AB, CD, EF, are here ſuppos'd by *Euclid* in one and the ſame Plane, otherwiſe the two preceding Demonſtrations wou'd be imperfect. But in *Prop.* 9. *l.* 11. we ſhall demonſtrate the Truth of this *Theorem*, tho' theſe three Lines be not in one and the ſame Plane.

USE.

This Propoſition may be of uſe to ſhew, that *if two right-Lines which cut each other, are parallel to two other Right-Lines, which interſect in the ſame Plane, theſe four Right-Lines contain two equal Angles.*

Fig. 54. As, if the two Lines AB, AC, are parallel to the two DE, DF, *viz.* AB to DE, and AC to DF, the two Angles A, D, are equal to each other.

PREPARATION.

Draw from the Point C taken at Pleaſure upon the Line AC, the right Line CG, parallel to the Line AB, and from the Point E taken at diſcretion upon the Line DE, the right Line EG, parallel to the Line AC; this Line EG will meet the firſt CG, in ſome Point, as G.

DEMONSTRATION.

Becauſe the two Lines GC, DE, are parallel to the ſame AB, the three AB, GC, DE, will be parallel to each other, as was juſt now demonſtrated, and in like manner becauſe the two Lines GE, DF, are parallel to the ſame AC, the three AC, GE, DF, will be parallel to each other. Wherefore *per Prop.* 29. all the alternate Angles, A, C, G, D, and conſequently the two A, D, will be equal to each other. *Which was to be prov'd.*

Tho'

Tho' the two Angles A, D, be not in the same Plane, they are, however, equal to each other, provided their Lines continue parallel each to each, as will be demonstrated in *Prop.* 10, 11. Plate 4. Fig. 54.

PROPOSITION XXXI.

PROBLEM X.

To draw thro' a given Point, a Right-Line parallel to a given Line.

TO draw thro' the given Point C, a Line parallel to the given Line AB; draw at pleasure thro' the given Point C, the Right-Line CD, which cuts the propos'd Line AB, in some Point as D, and make *per Prop.* 23. at the Point C, the Angle DCE equal to the Angle ADC, with the Right-Line CE, which will be parallel to AB. Fig. 55.

DEMONSTRATION.

The alternate Angles ADC, DCE, are equal *per constr.* therefore *per Prop.* 27. the Lines AB, CD, are parallel. *Which was to be done and demonstrated.*

USE.

The Use of Parallel-Lines is as frequent as that of Perpendiculars; it being certain that nothing can for Example be practis'd in *Perspective*, without drawing several Parallel-Lines, or which is the same thing, without drawing several Perpendiculars to the Ground-Line, because all Lines perpendicular to one and the same Line, are parallel to each other, as is evident *per Prop.* 28. In the description of Polar-Dials, the Hour-Lines are drawn Parallel to each other, and to the Sub-stile-Line, because these Sorts of Dyals have no Center at all, as we shall demonstrate in the *Dyalling. Fortification* cannot be without Parallel-Lines, when the Engineer wou'd draw the Ichnography of Parapets, Talus's, Esplanades, *&c.*

PROPOSITION XXXII.

THEOREM XXII.

Plate 4. Fig. 53.

In all Triangles, one of the Sides being extended, the exterior Angle is equal to the two interior opposite ones taken together; and the three Angles of a Triangle are together equal to two right Angles.

I Say, that if from the Triangle ABC, the Side AB be extended towards D, the exterior Angle CBD is equal to the two Interiors A, C, taken together; and that the three Angles A, ABC, C, are together equal to two right Angles.

PREPARATION.

Make *per Prop.* 23. at the Point B, the Angle DBE equal to the Angle A, with the Line BE, which will be parallel to the Line AC, *per Prop.* 28. and *per Prop.* 29. the Angle C will be equal to the Angle CBE.

DEMONSTRATION.

Since the Angle CBE is equal to the Angle C, and the Angle DBE to the Angle A, the two Angles A, C, taken together, will be equal to the two DBE, CBE, taken together, that is to say, to the whole exterior Angle CBD. *Which is one of the two things that was to be shewn.*

Since the exterior Angle CBD is equal to the two opposite interior A, C, if on each Side the Angle ABC is added, it will appear that the three Angles A, ABC, C, are together equal to the two ABC, CBD, that is to say, to two right Angles, *per Prop.* 13. *Which remain'd to be demonstrated.*

COROLLARY I.

It follows from this Proposition, that the three Angles of one Triangle are together equal to the three Angles taken together of another Triangle.

COROLLARY II.

Plate 4. Fig. 59.

If two Angles of one Triangle are equal to two Angles of another Triangle, each to each, the third Angle of the one will be equal to the third Angle of the other.

COROLLARY III.

In a Right-Angled Triangle, the two acute Angles taken together, are precisely equal to one right one.

COROLLARY IV.

Each Angle of an equilateral Triangle is 60 Degrees, because it is the third of two Right-Angles, which make 180 Degrees.

COROLLARY V.

All the Angles of a Polygon are equivalent to as many Times 180 Degrees, as the Polygon has Sides, except two, because it is divisible into so many Triangles. Whence it follows that in a Figure of four Sides, the four Angles make together four right ones, that is to say, 360 Degrees.

COROLLARY VI.

In all Polygons, each Side being extended, all the exterior Angles taken together are equal to four right ones, or to 360 Degrees. This results from this Proposition, and *Prop*. 13.

USE.

This Proposition is very useful in many Propositions of this and the following Books, and likewise in all Parts of Trigonometry, which considers a Triangle only with respect to its Angles, or its Sides. It is also very useful to measure upon the Ground an inaccessible Angle, as you have seen in the Use of *Prop*. 15. Engineers make great Use of it in raising Platforms, and they know that they have well measur'd the Angles of a Plan, when all the Angles of that Plan make together as many times 180 Degrees, as the Plan has Sides, except two.

PROP.

Plate 4. Fig. 56.

PROPOSITION XXXIII.

THEOREM XXIII.

The Right-Lines are equal and parallel, which join the Extremities, lying the same way, of two other equal and parallel right Lines.

I Say, that if the two Right-Lines AB, CD, are parallel and equal, the Right-Lines AC, BD, which join their extremities, are also parallel and equal.

DEMONSTRATION.

If the Right-Line AD, be drawn, it will be known *per Prop.* 4. that the two Triangles ADB, ADC, are equal to each other, because they have the common Side AD, the Side AB equal to the Side CD, *per Sup.* and the included Angle ADC equal to the included Angle BAD, *per Prop.* 29. Wherefore the Line AC will be equal to the Line BD: *Which is one of the two Things which was to be shewn*: And the Angle DAC will be equal to the Angle ADB, wherefore *per Prop.* 27. the two Lines AC, BD, will be parallel to each other. *Which remain'd to be shewn.*

USE.

This Proposition serves for the Demonstration of *Prop.* 35. and also to measure upon the Ground an accessible Line at its two Extreams, and inaccessible at its Middle, as we shall teach in our *Practical Geometry*.

PROPOSITION XXXIV.

THEOREM XXIV.

In all Parallelograms, the Angles and the opposite Sides are equal to each other, and the Diagonal divides it equally in two.

Fig. 56. I Say, that if the Figure ABDC be a Parallelogram, the opposite Angles B, C, are equal to one another, as well as the two BAC, BDC: and in like manner the

the opposite Sides AB, CD, are equal to one another, as well as the two AC, BD: And lastly, the Diagonal AD divides the Parallelogram ABDC equally in two; that is to say, the two Triangles ADB, ADC, are equal to one another.

DEMONSTRATION.

Because the two Lines AB, CD, are Parallels *per Sup.* the two alternate Angles BAD, ADC, will be equal to one another, *per Prop.* 29. as well as the two alternate Angles ADB, DAC, by reason of the two Parallels AC, BD. From whence it follows, *per Prop.* 32. that the third Angle B will be equal to the third Angle C, and *per Ax.* 2. the whole Angle BAC, equal to the whole Angle BDC. *Which is one of the three Things which was to be demonstrated.*

Since therefore the two Triangles ADB, ADC, are equiangular, and that they have the common Side AD, similarly posited, they will be equal to one another *per Prop.* 26. *Which is the second of the three Things that was to be shewn.*

Lastly, The Sides opposite to the equal Angles of the two equal Triangles ADB, ADC, to wit, AB, CD, and AC, BD, will be equal to each other. *Which remain'd to be prov'd.*

USE.

The Method which you will find in our *Practical Geometry*, to measure the Height and Bigness of a Mountain, by the means of a Plomb-Line, and a long Rule, which is call'd *Cultellation*, is founded upon this Proposition; the which serves likewise for the Division of a Field, when it is a Parallelogram, at least when you wou'd divide it equally in two, which is done by the Diagonal AD, when you have no determin'd Point to make that Division. But if you wou'd divide it equally in two, by a Right-Line drawn from a Point given in one Side, as through the Point E, you must draw from this Point E, through the Point F, the middle of the Diagonal AD, the Right-Line EFG, which will divide the Parallelogram ABDC into two equal Trapeziums ACGE, EGDB, by reason of the Triangle AFE equal to the Triangle DFG, *per Prop.* 26. and by reason of the two equal Trapeziums, CF, BF, *per Ax.* 3. because *per Prop.* 34. the two Triangles ADB, ADC, are equal to one another. Fig. 57.

It

Plate 4. Fig. 57. It is known that a Quadrangular Field is a Parallelogram, when of its four Angles, the two opposite are equal, or when of its four Sides the two opposite are equal, as it is easy to demonstrate *per Prop.* 8. Which discovers the Original and Demonstration of a certain Instrument, commonly made use of to draw parallel Lines, and which upon that account is call'd a *Parallel Ruler*, because it is compos'd of two long Rulers fastned together by two other lesser Rulers, and equal to one another, which preserve the two great Rulers always in a parallelism whatever Situation you give them.

Wherefore when you wou'd by the help of this Instrument draw thro' a given Point, a Line parallel to a given Line, there is nothing more to do than to apply the Edge of one of the two Rulers along the given Line, and the second Ruler being kept steady and immoveable, you must advance the first as far as the given Point, to the end that thro' that Point you may draw along the Ruler a Right-Line, which will be parallel to the propos'd one.

This Proposition serves also to demonstrate *Euclid's* eleventh *Axiom*, which we shall prove in the following manner, being a Demonstration that seems to me very plain and very natural.

Fig. 58. I say then, that if the two Right-Lines AB, CD, are intersected by a third Right-Line EF, so that the two interior Angles BEF, EFD, which are on the same Side, are together less than two right ones; the two Lines AB, CD, being extended, will meet on this same Side.

DEMONSTRATION.

To demonstrate this Truth, it will suffice to have demonstrated, that if on the same Side with the interior Angles BEF, EFD, you draw the Right-Line GH parallel to the Line EF, and terminated by the two Lines AB, CD, this Line GH, will be less than the Line EF.

For this purpose draw thro' the Point H, the Right-Line HI, parallel to the Line AB. It is evident that this Line HI, meets the Line EF, at the Point I, between the Points E, F, because if it meet it beyond the Point F, as in L, it wou'd follow that the two Angles BEF, HLF, wou'd be together equal to two right ones, *per Prop.* 29. and consequently greater than the two BEF, EFD, which are suppos'd less together than two right ones, and that so by taking away the common Angle BEF, the Angle HLE, wou'd remain greater than the Angle EFD, which is impossible, because the Angle EFD, being

being exterior, is greater than the interior opposite one HLF, *per Prop.* 16. the same Point I, cannot also fall on the Point F, because the Lines AB, CD, wou'd be Parallels, and so the two interior Angles BEF, EFD, wou'd together be equal to two right ones, *per Prop.* 28. and yet they are suppos'd less. Therefore since the Point I, falleth between the two Points E, F, and that the Figure GHIE is a Parallelogram, whereof the opposite Sides GH, EI, are equal, *per Prop.* 34. it follows that the Line GH is less than the Line EF. *Which was to be demonstrated.* Plate 4. Fig. 58.

PROPOSITION XXXV.

THEOREM XXV.

Parallelograms are equal to one another, when they have the same Base, and are between the same Parallels.

I Say, that the Parallelograms EFGH, EIKH, are equal to one another, because they are between the two Parallels AB, CD, and have the common Base EH.

DEMONSTRATION.

The Sides IK, FG, are equal each to the Side EH, *per Prop.* 34. and *per Ax.* 1. they are equal to one another; and if the Side GI be added to them, you will have *per Ax.* 2. the Side FI, of the Triangle FEI, equal to the Side GK, of the Triangle GHK; and because the Side EF is equal to the Side GH, and the Side EI equal to the Side HK, *per Prop.* 34. it follows *per Prop.* 8. that the two Triangles EFI, GHK, are equal to one another; wherefore if from each the common Triangle GLI, be taken away, there will remain the Trapezium FL, equal *per Ax.* 3. to the Trapezium KL, and lastly if to each of these two equal Trapezia FL, KL, the Triangle ELH, be added, you will have the Parallelogram EFGH equal *per Ax.* 2. to the Parallelogram EIKH. *Which was to be prov'd.* Plate 4. Fig. 59.

SCHOLIUM.

This Theorem may be demonstrated more easily by the *Method of Indivisibles* in this manner. Imagine the Parallelogram EFGH, to be divided into as many little equal Parallelograms as you please, by Lines parallel to one another, and to the common Base EH, to which they will be all

Plate 4. Fig. 60. equal, and consequently equal to one another, these Lines being continued, will divide the other Parallelogram EIKH, in so many Parallelograms equal to each other, and to the preceding ones; which makes that these two Parallelograms EFGH, EIKH, are equal to one another, because whatever Division is made, there will still be as many Lines of the same Length, and equally close, in the one as in the other: So that if the Division be infinite, as it is still suppos'd to be, which occasion'd the Name of the *Method of Indivisibles* to be given this sort of Demonstration, each Parallelogram will be compos'd of an equal Number of equal Lines, that is to say, of little equal Parallelograms whereof the Breadth is infinitely little, and consequently they will be equal to one another. *Which was to be shewn.*

This Method of Indivisibles is of great Use to demonstrate the hardest Theorems in *Geometry*, principally for the Tangents of curved Lines, and for the *Quadrature* of Curves, that is to say, to reduce a Curvilineal Figure into a Rectilineal one; it being certain, that by means thereof Theorems may be demonstrated, which wou'd be difficult to be done by *Euclid*'s Elements alone. You will find an Example of it in the first Theorem of our *Planimetry*.

The most Learned Men allow of the Geometry of Indivisibles, and none but those who are less expert reject it and that doubtless because they are easily mistaken, by not knowing how to make a just Application of it, for want of well understanding the Foundation of this Geometry, which consists principally in taking for the Area of a Surface, the Sum of the infinite Lines which fill it, and for the Solidity of a Body, the infinite Surfaces it is compos'd of; so that two Surfaces are esteem'd equal, when each is fill'd with an equal Sum of Lines, in like manner equal and parallel to each other; and likewise two Solids are esteem'd equal, when the one and the other is compos'd of an equal Sum of Surfaces, in like manner equal and parallel to each other, *&c.*

USE.

Fig. 61. This Proposition serves for the Demonstration of the following and several others, and likewise to measure a Parallelogram, which is not Rectangular, as EIKH, because it may be reduc'd into another which is Rectangular, to wit, in drawing from the two Extremities E, H, of the Side EH, the two Lines EF, GH, perpendicular to the Side EH, which being terminated by the other opposite and parallel Side IK, extended as far as shall

ſhall be neceſſary, will finiſh the Rectangular Parallelogram EFGH, equal to the propos'd Parallelogram EIKH, the Area whereof will conſequently be known, if you multiply together the two Sides EF, EH, which form the Right-Angle E: as if EF is for Example 5 Feet, and EH 3, by multiplying 5 by 3, you will have 15 ſquare Feet, for the Content of the Rectangular Parallelogram EFGH, or of its equal EIKH. Plate 4. Fig. 61.

PROPOSITION XXXVI.

THEOREM XXVI.

Parallelograms are equal to each other, when they have equal Baſes, and are between the ſame Parallels.

I Say, that if the two Parallelograms EFGH, IKLM, are between the ſame Parallels AB, CD, and that their Baſes EH, IM, be equal to each other, theſe Parallelograms EFGH, IKLM, are alſo equal to each other. Fig. 62.

PREPARATION.

Join the two Extremities of the two equal and parallel Baſes EH, KL, by the Right-Lines EK, HL, which will be alſo equal and parallel, *per Prop.* 33. ſo that *per Def.* 34. the Figure EKLH will be a Parallelogram.

DEMONSRATION.

Since each of the two Parallelograms EFGH, IKLM, is equal to the Parallelogram EKLH, it follows *per Ax.* 1. that they are equal to each other. *Which was to be ſhewn.*

SCHOLIUM.

This Propoſition is virtually the ſame as the preceding, becauſe to have one and the ſame Baſe is the ſame thing as to have equal Baſes; and it is expreſs'd more generally in *Prop.* 1. 6.

When it is ſaid, that *two Parallelograms are between the ſame Parallels*; it ſignifies that two of their oppoſite Sides do meet in two Lines parallel to each other; ſuch as AB, CD, in this Place.

Plate 4. Fig. 63.

USE.

This Proposition is very serviceable to divide in as many equal Parts as you will, a Field which hath the Figure of a Parallelogram, as if you wou'd divide in three equal Parts, for Example, the Parallelogram ABCD, you must divide two of its opposite Sides AD, BC, each in three equal Parts, and you must join the opposite Points of Division by the Right-Lines EG, FH, which will divide the propos'd Parallelogram ABCD, in three less Parallelograms, which will be equal to each other, since their Bases are equal to each other.

PROPOSITION XXXVII.

THEOREM XXVII.

Triangles are equal, when they have the same Base, and are between the same Parallels.

Fig. 64.

I Say, that if the Triangles EFG, EFH, have the same Base EF, and are inclos'd between the same Parallels AB, CD, so that their Vertex's G, H, do terminate at the same Line AB, parallel to the common Base EF; these two Triangles EFG, EFH, are equal to each other.

PREPARATION.

Take upon the Line AB, the Lines GA, HB, equal each to the common Base EF, and join the Right-Line AE, which will be Parallel to the Line FG, *per Prop.* 33. and the Line BF, which will be likewise parallel to the Line EH.

DEMONSTRATION.

Since the Side EG, of the Triangle EFG, is the Diagonal of the Parallelogram EFGA, this Triangle EFG, will be the half of the Parallelogram EFGA, *per Prop.* 34. and by the same Reason the Triangle EFH, will be the half of the Parallelogram EFBH; and as the Parallelograms EFGA, EFBH, are equal to each other, *per Prop.* 35. their Halves, that is to say, the Triangles EFG, EFH, will be also equal to each other. *W. W. D.*

USE.

USE.

This Poposition serves to demonstrate that *when two Right-Lines intersect between two Parallels, their Parts are proportional*; as if the two Lines EH, FG, intersect at the Point I, between the two Parallels AB, CD, their Parts IE, IH, IF, IG, are proportionable; for if the Right-Lines EG, FH, be join'd, it will be known *per Prop.* 37. that the two Triangles EFG, EFH, are equal to each other, therefore if from each you substract the common Triangle EIF, there will remain *per Ax.* 3. the Triangle EIG, equal to the Triangle FIH, and by reason of the two equal Angles EIG, FIH, *per Prop.* 15. it follows *per* 15. 6. that the four Lines IE, IH, IF, IG, are proportional. *Which was to be demonstrated.* Plate 4. Fig. 65.

This Proposition is also very serviceable, to reduce any right lin'd Figure into a Triangle, which is done thus,

First of all, to reduce into a Triangle the Trapezium ABCD, having drawn at pleasure the Diagonal BD, draw from the Angle C, opposite to that Diagonal, the Right-Line CE, parallel to the same Diagonal, BD, and from the Point E, where it meets the extended Side AB, draw to the Angle D, the Line DE, and the Triangle ADE, will be equal to the propos'd Trapezium ABCD. Plate 5. Fig. 66.

DEMONSTRATION.

Since the two Triangles DCB, DEB, have the same Base BD, and are between the same Parallel BD, CE, they will be equal to each other, *per Prop.* 37. Wherefore if from each the common Triangle BFD, be put away, there will remain *per Ax.* 3. the Triangle CFD, equal to the Triangle BEF, whereof each being added to the Trapazium ABFD, there will be had *per Ax.* 2. the Trapezium ABCD, equal to the Triangle ADE. *Which was to be done and demonstrated.*

'Tis in the same manner, that a Figure of more than four Sides is reduc'd into a Triangle, to wit, by reducing it first into another which hath a Side less, as you have just now seen, and this into another, which has likewise a Side less, and so on, until you come to a Triangle.

Plate 5. Fig 67.

PROPOSITION XXXVIII.

THEOREM XXVIII.

Triangles are equal when they have equal Bases, and are between the same Parallels.

I Say, that the two Triangles EFG, HIK, which are between the same Parallels AB, CD, and whereof the Bases EF, HI, are equal to each other, are also equal to each other.

PREPARATION.

Take upon the Line AB, the Line GA, equal to the Base EF, and join the Right-Line AE, which will be parallel to the Side FG, *per Prop.* 33. Take upon the same Line AB, the Line KB, equal to the Base HI, and join the Line BI, which will be parallel to the Side HK.

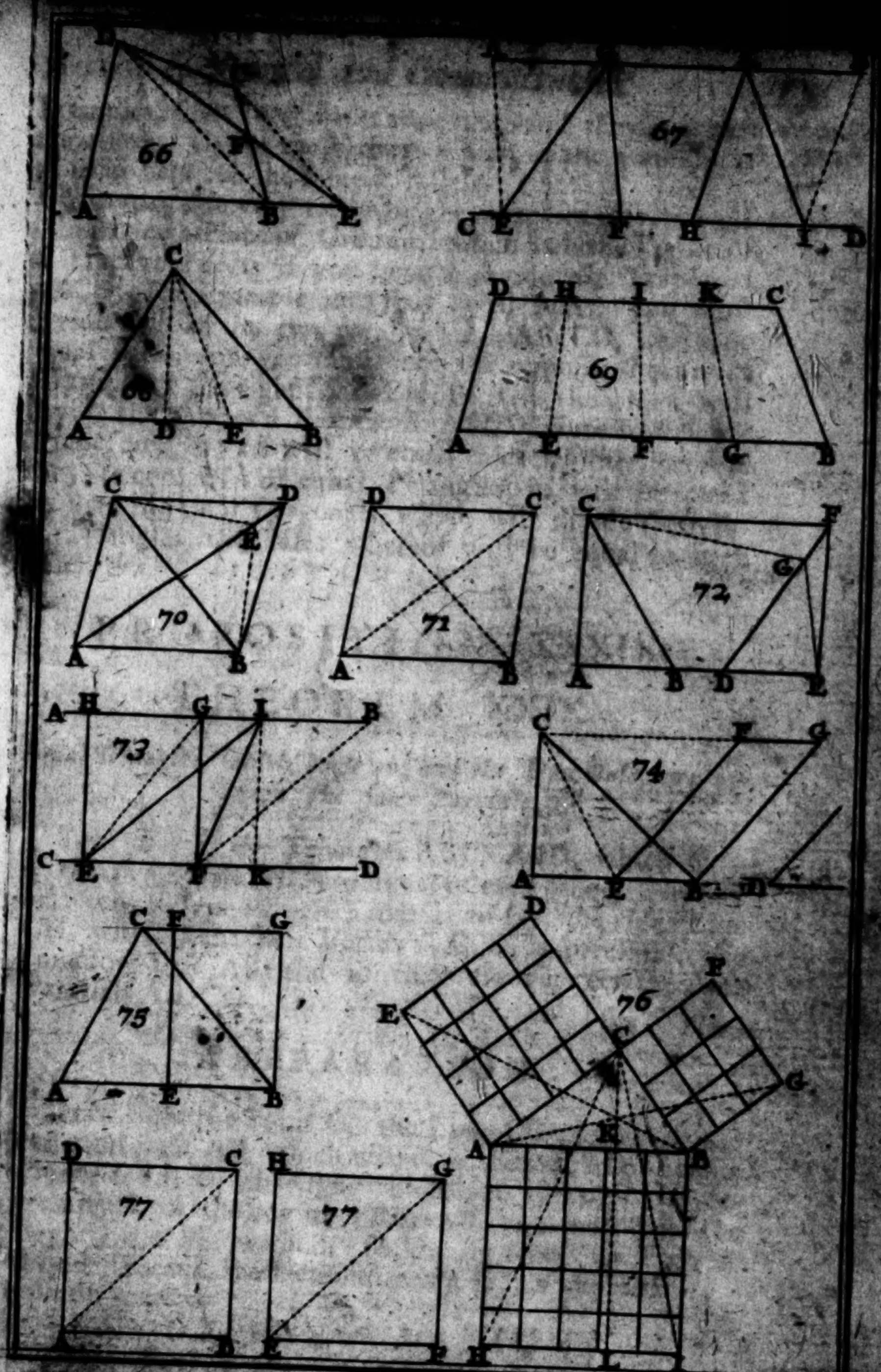
DEMONSTRATION.

Because the Side EG is the Diagonal of the Parallelogram EFGA, the Triangle EFG, will be the half of that Parallelogram *per Prop*. 34. and in like manner, since the Side IK is the Diagonal of the Parallelogram HIBK, the Triangle HIK, will be the half of that Parallelogram; and as the two Parallelograms EFGA, HIBK, are equal to each other *per Prop*. 36. it follows that their halves, that is to say the Triangles EFG, HIK, are also equal to each other. *Which was to be shewn.*

USE.

This Proposition serves to divide a Triangular Field into as many equal Parts as you will, by right Lines drawn from one of its Angles thus.

Fig. 68. To divide the Triangle ABC, for example into three equal Parts, by right Lines drawn from the Angle C, divide the Side AB, opposite to this Angle C, into three equal Parts at the Points D, E, and draw thro' these Points E, D, to the Angle C, as many right Lines, which will divide the propos'd Triangle ABC into three equal Triangles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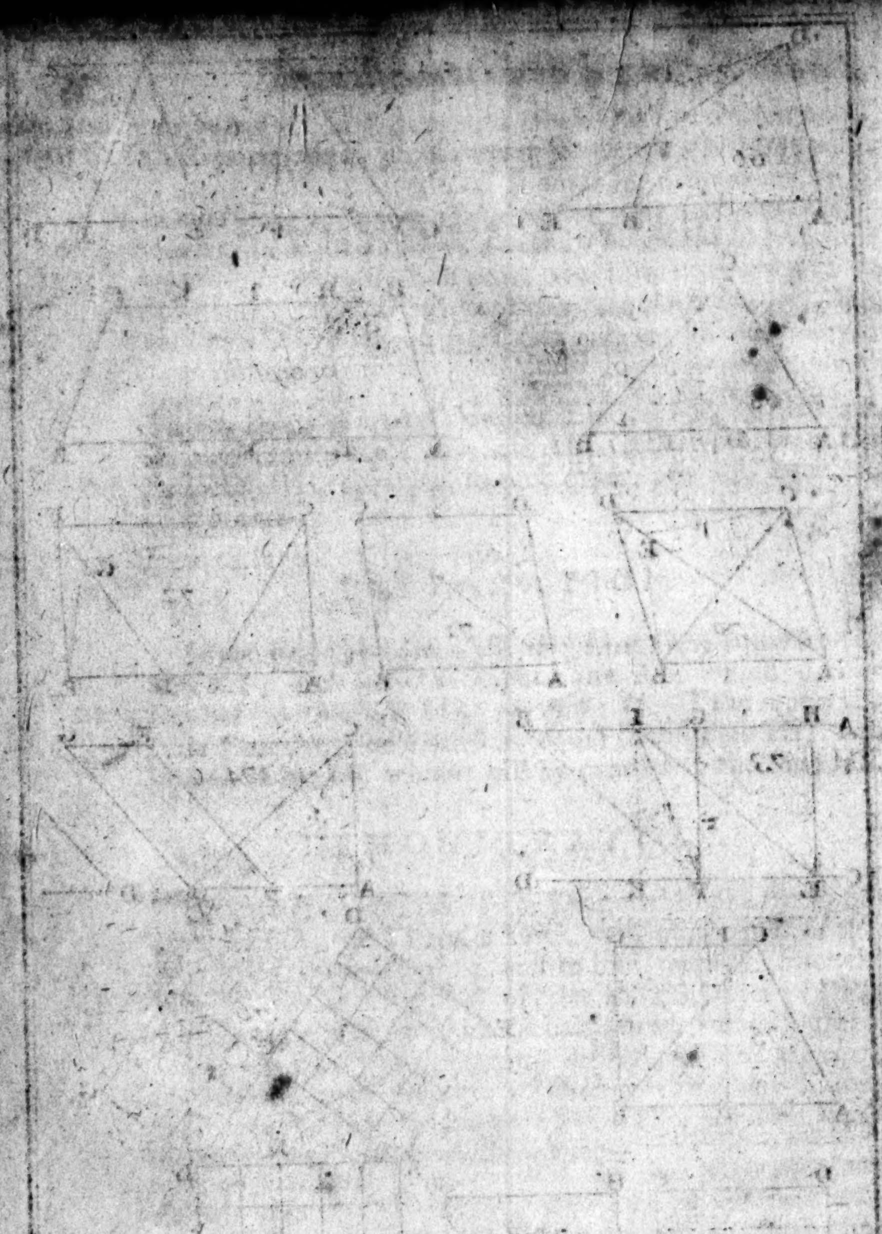


Triangles, since their Bases are equal, and they have the same Point C, for the Vertex, which is the same thing as to be between the same Parallels. Plate 5. Fig 68.

You may also very easily by the means of this Proposition divide a Piece of Ground, which hath the Figure of a Trapezoid; as if you wou'd divide the Trapezoid ABCD, for Example into four equal Parts, you must divide each of its two parallel Sides AB, CD, into four equal parts, and you must join the opposite Points of Division by the right Lines EH, FI, GK, the which will divide the propos'd Trapezoid ABCD, into four lesser Trapezoids, which will be equal to each other, because they are compos'd of equal Triangles, as will be found by drawing their Diagonals, which will divide them into Triangles, the Bases whereof will be equal to each other, &c. Fig. 69.

PROPOSITION XXXIX.

THEOREM XXIX.

Equal Triangles, which have one and the same Base, are between the same Parallels.

I Say, that if the Triangles ABC, ABD, which have the same Base AB, are equal to each other, they are between *the same Parallels*; that is to say, the right Line CD, which joins their Vertex's, C, D, is parallel to the common Base AB, and so they are between the same Parallels AB, CD. Fig. 70.

PREPARATION.

Draw from the Point C, the Line CE parallel to the common Base AB, which will meet the Side AD, in some Point, as in E, through which, and through the extremity B, of the common Base AB, you must draw the Right-Line BE, without considering where the Point E falleth, because the Demonstration is still the same.

DEMONSTRATION.

Since the Triangles ABC, ABE, are between the same Parallels AB, CE, *per constr.* and have the common Base AB, they will be equal to each other, *per Prop.* 37. and as the Triangle ABD, is equal to the Triangle ABC, *per Sup.* it follows *per Ax.* 1. that the two Triangles ABD,

Plate 5. Fig. 70. ABE, are equal to each other, and *per Ax.* 8. the Point E falleth upon the Point D, and the Line CE, upon the Line CD, and consequently the Line CD, is parallel to the common Base AB. *Which was to be shewn.*

USE.

Fig. 71. This Proposition serves to demonstrate that, *Every Quadrilateral which is divided equally in two by each of its two Diagonals, is a Parallelogram*; that is to say, that if the Quadrilateral ABCD, be divided equally in two, by the Diagonal AC, and also equally in two by the other Diagonal BD, so that the three Triangles ABC, ABD, ACD, be equal to each other, *per Ax.* 7. as being each the half of the Quadrilateral ABCD; this Quadrilateral ABCD, will be a Parallelogram.

DEMONSTRATION.

Since the two Triangles ABC, ABD, which have the same Base AB, are equal to each other *per Sup.* they will be between the same Parallels *per Prop.* 39. that is to say, that the Line CD, will be parallel to the common Base AB. It will be known in the same manner, that by Reason of the two equal Triangles ACB, ABD, which are upon the same Base AD, the Line BC is parallel to the common Base AD, and so the Figure ABCD is a Parallelogram. *Which was to be shewn.*

PROPOSITION XL.

THEOREM XXX.

Equal Triangles, which have equal Bases upon one and the same Right-Line, are between the same Parallels.

Fig. 72. I Say, that if the equal Triangles ABC, DEF, have their equal Bases AB, DE, upon the Right-Line AE, their Vertex's C, F, are terminated by the Right-Line CF, parallel to the first Right-Line AE.

PREPARATION.

Draw from the Point C, the Line CG, Parallel to AE, which will meet the Side DF, in some Point, as

as in G, thro' which, and thro' the Extremity E, of the Base DE, you must draw the Right-Line GE, without considering where the Point G falleth, because the Demonstration will be always the same, as you shall see. Plate 5. Fig. 72.

DEMONSTRATION.

Since the Triangles ABC, DEG, are between the same Parallels AE, CG, *per constr.* and that their Bases, AB, DE, are equal, *per. Supposition,* they will be equal to each other, *per Prop.* 38. and as the Triangle DEF, is suppos'd equal to the Triangle ABC, it follows *per Ax.* 1. that the two Triangles DEF, DEG, are equal to each other, and *per Ax.* 8. the Point G, falleth on the Point F, and the Line CG upon the Line CF, and consequently the Line CF, is parallel to the Line AE, because the Line CG, hath been suppos'd parallel to the same Line AE. *Which was to be shewn.*

PROPOSITION XLI.

THEOREM XXXI.

If a Parallelogram and a Triangle have one and the same Base, and are between the same Parallels, the Parallelogram will be double the Triangle.

I Say, that if the Parallelogram EFGH, and the Triangle EFI, have one common Base EF, and are between the same Parallels AB, CD, so that the Vertex I, of the Triangle EFI, terminates precisely at the Line AB, parallel to the common Base EF; the Parallelogram EFGH, is double the Triangle EFI. Fig. 73.

PREPARATION.

Draw from the Extremity F, of the common Base EF, the right Line FB, parallel to the Side EI, of the Tri- EFI, and you'll have the Parallelogram EFBI.

DEMONSTRATION.

Because the Parallelogram EFGH is equal to the Parallelogram EFBI, *per Prop.* 35. and that the Parallelogram EFBI is double the Triangle EFI, *per Prop.* 34. it

Plate 5. Fig. 73. it follows that the Parallelogram EFGH, is also double the Triangle EFI. *Which was to be shewn.*

SCHOLIUM.

This Proposition may be demonstrated otherwise, and very easily, if instead of drawing the Parallel FB, you draw the Diagonal EG, for then it will be known *per Prop.* 37. that the Triangle EFG is equal to the Triangle EFI; from whence it follows that the Parallelogram EFGH, being double the Triangle EFG, *per Prop.* 34. it is also double the Triangle EFI. *Which was to be shewn.*

USE.

This Proposition serves as a Lemma to the following, and also to demonstrate *Prop.* 47. It is the Foundation of the Method, generally made use of to find out the Area of a Triangle, which is to multiply the Base of the Triangle by its perpendicular drawn from the opposite Angle, and to take the half of the Product; because by multiplying the Base EF, of the Triangle EFI, by its Perpendicular IK, you have the Contents of a Rectangular Parallelogram, as EFGH wou'd be, which is double the Triangle, as we have just now demonstrated, which makes that the half of it is taken, to have the Area of a Triangle.

PROPOSITION XLII.

PROBLEM XII.

To describe a Parallelogram equal to a Triangle given, and having an Angle equal to a given right-lin'd Angle.

Fig. 74. TO reduce the given Triangle ABC, into a Parallelogram, which hath one Angle equal to the given Angle D, divide its Base AB, equally in two at the Point E, *per Prop.* 10. and *per Prop.* 31. draw thro' the Angle C, opposite to the Base AB, the indefinite Right-Line CG, parallel to the same Base AB. Make by *Prop.* 23 at the Point E, the Angle BEF, equal to the given D, and *per Prop.* 31. draw thro' the Point B, the Right-Line BG, parallel to the Line EF; and the Parallelogram EBGF, will be equal to the propos'd Triangle ABC.

DE-

Plate 5. Fig. 74.

DEMONSTRATION.

If you join the Right-Line CE, it will be known *per Prop.* 38. that the two Triangles CEA, CEB, are equal to each other, and that consequently the Triangle ABC, is double the Triangle CEB; and as the Parallelogram EFGB, is also double the Triangle CEB, *per Prop.* 41. it follows *per Ax.* 6. that the Parallelogram EFGB, is equal to the Triangle ABC. *Which was to be done and demonstrated.*

USE.

This Proposition serves as a Lemma to the following, and also to reduce a Triangle into a Rectangular Parallelogram, which will be done if you draw the Line EF, perpendicular to the Base AB. From whence is deriv'd the common method of finding the Area of a Triangle, as of the Triangle ABC, which is to multiply the half BE, of its Base AB, by the Perpendicular EF, which is equal to the Perpendicular which wou'd fall from the Angle C, upon the Base AB, for thus you have the Area of the Rectangular Parallelogram EFBG, which hath been demonstrated equal to the Triangle ABC. Fig. 75.

We leave out here, Prop. XLIII. XLIV. *because we can do without 'em in the Resolution of what is to follow, and because they are not of any considerable use in Geometry.*

PROPOSITION XLV.

PROBLEM XIII.

To describe a Parallelogram equal to a right-lin'd Figure given, and having an Angle equal to a given Angle.

IT is evident that if *per Prop.* 37. you reduce the given Rectiline into a Triangle, and that Triangle into a Parallelogram, which hath an Angle equal to the given one, *per Prop.* 42. the Problem will be resolv'd. Plate 5. Fig. 76.

PRO-

Plate 5.
Fig. 71.

PROPOSITION XLVI.

PROBLEM XIV.

To describe a Square upon a given Line.

TO describe a Square upon the given Line AB, draw *per Prop.* 11. from the two Extremities A, B, the two Lines AD, BC, equal and perpendicular each to the same Line AB, and join the Right-Line CD, and the Figure ABCD will be a Square, so that its four Angles will be right ones, and its four Sides equal to each other.

DEMONSTRATION.

Since the two Lines AD, BC, are equal each to the same AB, *per constr.* they will be equal to each other *per Ax.* 1. and because they have been made perpendicular to the same AB, they will be parallel to each other, *per Prop.* 28. and *per Prop.* 33. the two Lines AB, CD, will be equal and parallel to each other. Thus the four Sides of the Figure ABCD, will be equal to each other. *Which is one of the two things to be shewn.*

Since the Figure ABCD, is a Parallelogram, as we have just now discover'd, the Angle C will be equal to its opposite A, *per Prop.* 34. and consequently a right one, and likewise the Angle D, will be equal to its Opposite B, and consequently a right one. Thus the four Angles of the Figure ABCD, will be right ones. *Which remain'd to be prov'd.*

USE.

This Proposition serves as a Lemma to the following Theorem, and serves also for the Demonstration of almost all the Popositions of the second Book, and upon many other Occasions.

PRO-

Plate 5. Fig. 76.

PROPOSITION XLVII.

THEOREM XXXIII.

In Right angled Triangles the Square of the Hypotenuse is equal to the Sum of the Squares of the two other Sides.

I Say, that the Square ABIH, describ'd upon the Hypotenuse, or upon the Side AB, opposite to the Right-Angle C, of the Rectangular Triangle ABC, is equal to the Sum of the Squares ACDE, BCFG, describ'd on the two other Sides AC, BC.

PREPARATION.

Draw from the Right-Angle C, the Line CKL perpendicular to the Hypotenuse AB, and join the right Lines CH, CI, and AG, BE; for I suppose that *per Prop.* 46. a Square hath been describ'd upon each of the three Sides of the Rectangular Triangle ABC, whereof the Hypotenuse AB, is here suppos'd 5 Feet, the Side AC, 4. and the other Side BC, 3. and then 'tis already seen by Experience, that the Square alone of the Hypotenuse AB, hath as many Square Feet, to wit, 25. as the two other Squares contain together, for the Square of AC, contains 16, and the Square of BC, contains 9, which with 16, make 25. Let us see at present the

DEMONSTRATION.

The two Triangles ABG, BCI, are equal to each other, *per Prop.* 4. because they have the two Sides AB, BG, equal to the two BI, BC, and the compriz'd Angle ABG, equal to the compris'd Angle CBI, each being compos'd of a Right-Angle, and of the common Acute-Angle ABC.

In like manner the two Triangles ABE, ACH, are equal to each other, because they have the two Sides AB, AE, equal to the two AH, AC, and the compris'd Angle CAH, equal to the compris'd Angle BAE, each being

Plate 5. Fig. 76. being compos'd of a Right-Angle, and of the common Acute-Angle BAC.

Becauſe the two Angles ACB, ACD, are right ones, and conſequently equal together to two right ones, it will be known *per Prop.* 14. that BCD is a right Line, and by the ſame Reaſon, it will be known that ACF, is a Right-Line, by reaſon of the two Right-Angles BCA, BCF.

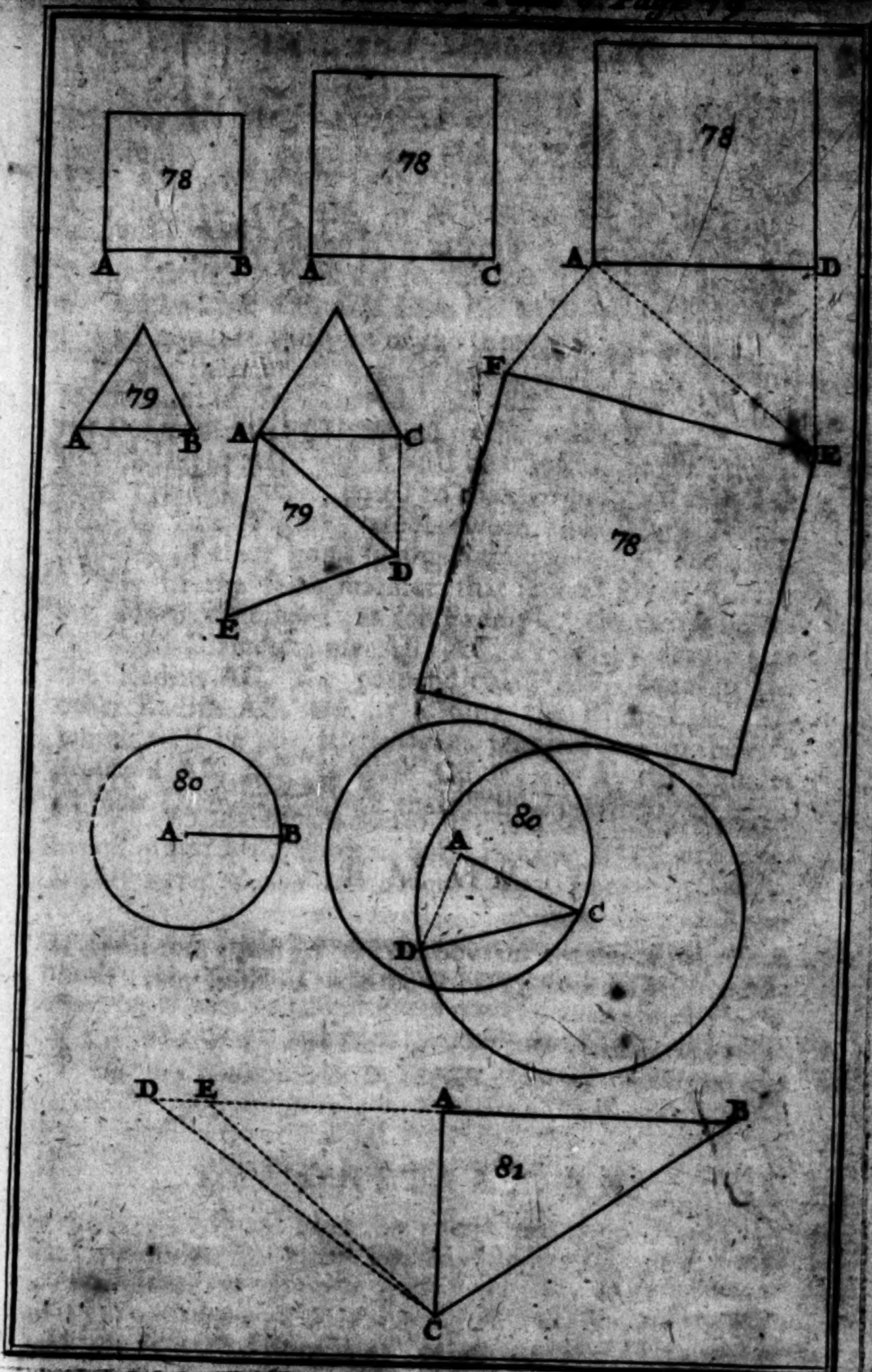
Becauſe the Triangle ABG, and the Parallelogram BCFG, have the ſame Baſe BG, and are between the ſame Parallels AF, BG, the Parallelogram BCFG, will be double the Triangle ABG, *per Prop.* 41. It will be known in the ſame Manner, that the Parallelogram KLIB, is double the Triangle BCI, becauſe they have the ſame Baſe BI, and are between the ſame Parallels CL, BI. From whence it is eaſy to conclude, that as each of the two Triangles, ABG, BCI, which have been demonſtrated equal, is the half of its Parallelogram, as it hath been demonſtrated; their doubles, to wit, the Square BCFG, and the Parallelogram KLIB, are equal to each other.

It may be demonſtrated in the ſame Manner, that the Square ACDE, is equal to the Parallelogram AKLH, from whence it follows that the Sum of the two Parallelograms BKLI, AKLH, that is to ſay, the ſingle Square ABIH, is equal to the Sum of the two Squares BCFG, ACDE. *Which was to be demonſtrated.*

This Demonſtration ſuppoſes that the Line CKL, is parallel to each of the two BI, AH, which is evident *per Prop.* 28. becauſe each of thoſe three Lines is *per conſtr.* perpendicular to the ſame Line AB.

USE.

This Propoſition ſerves not only for the Demonſtration of the following, and of many others in the ſucceeding Books, but it ſerves alſo as a Foundation to a great Part of the Mathematicks. You will ſee the Uſe of it in Trigonometry, for the Conſtruction of the Table of Sines, Tangents, and Secants; and we will here teach the Uſe of it, for the Addition of Squares, and of other regular Figures, the Sides whereof and the Angles are equal, and alſo for the Addition of Circles.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To find a Square equal to the Sum of the three given Squares AB, AC, AD, draw to the Side AD, the Perpendicular DE, equal to the Side AC, and join the right Line AE, which will be the Side of a Square equal to the two Squares AD, DE, or AC, by reason of the Right-Angle D: Wherefore if you draw to the Side AE, the Perpendicular AF, equal to the last Side AB, and join the Right-Line EF, this Line EF will be the Side of a Square equal to the Sum of the three AB, AC, AD. Plate 6. Fig. 78.

In like manner to find an Equilateral Triangle equal to the Sum of the two AB, AC, draw to the Side AC, the Perpendicular CD, equal to the other Side AB, and join the right AD, which will be the Side of the Equilateral Triangle ADE, equal to the two propos'd AB, AC, because like Figures are between themselves as the Squares of their homologous Sides. *per* 20. 6. See 31. 6. Fig. 79.

'Tis in the same manner that several given Circles are added together; as for Example, the two whereof the Semi-Diameters are AB, AC, to wit by drawing to the Radius AC, the perpendicular AD, equal to the other Radius AB, and by joining the Right-Line CD, which will be the Radius of a Circle equal to the two propos'd AB, AC, because Circles are as the Squares of their Diameters, or of their Semi-diameters, *per* 2. 12. Fig. 80.

LEMMA.

If upon two equal Lines two Squares are describ'd, those two Squares will be equal to each other.

I Say, that if the two Sides AB, EF, are equal to each other, the two Squares ABCD, EFGH, are also equal to each other. Plate 5. Fig. 77.

DEMONSTRATION.

If you draw the two Diagonals, AC, EG, they will divide their Squares equally in two, per Prop. 34. *in such manner that the Triangle ABC, will be the half of the Square ABCD, and the Triangle EFG, the half of the Square EFGH; and because these two Triangles ABC, EFG, are equal to each other,* per Prop. 4. *it follows that their Doubles, that is to say, the Squares ABCD, EFGH, are also equal to each other.* Which was to be shewn.

PROPO-

Plate 6. Fig. 81.

PROPOSITION XLVIII.

THEOREM XXXIV.

If in a Triangle the Square of one Side be equal to the Sum of the Squares of the two other Sides, the Angle opposite to that Side is a right one.

I Say, that if the Square of the Side BC, of the Triangle ABC, be equal to the Sum of the Squares of the two other Sides AB, AC, the Angle A, opposite to the first Side BC, is a right one.

PREPARATION.

Draw *per Prop.* 11. the Line AD, perpendicular to AC and equal to the Side AB, and join the right CD.

DEMONSTRATION.

By reaſon of the Right-Angle CAD, the Square of the Side CD is equal to the Square of the two other Sides AC, AD, of the Rectangular Triangle DAC, *per Prop.* 47. and becauſe the Side AB is equal to the Side AD, *per conſtr.* the Square of AB, will be equal to the Square of AD, *per preceding Lemma*. Thus the Square of CD, will be equal to the Sum of the Squares of AB, AC, and as this Sum is equal to the ſquare of BC, *per Sup.* it follows that the ſquare of CD, is equal to the Square of CB, and that conſequently the two Sides CD, CB, are equal to each other. Wherefore *per Prop.* 8. the Triangles ADC, ABC, will be equal to each other, and the Angle CAB will be equal to the Angle CAD, and conſequently a right one. *Which was to be ſhewn.*

USE.

This Propoſition, which is the Inverſe of the Preceding, ſerves to draw a Perpendicular through the Extremity of a Line given upon the Ground, as A, of the given Line AD, thus, Take from A, as far as E, upon the given Line AD, the Length of four Yards, and faſten at the Point A, a Cord 3 Yards long, and at the Point E, another Cord 5 Yards long. It is evident *per Prop.*

Prop. 48. that if you stretch the two Cords, and join together their Extremities, you will have the Point C of the Perpendicular AC, because 3, 4, 5, makes in Numbers a Rectangular Triangle. Plate 6. Fig. 81.

Instead of 3 Yards for AC, you may measure it 5, and instead of 4 for AE you may take 12; and then instead of 5, you must take 13, for the Cord, or Hypotenuse CE, because 5, 12, 13, is a Rectangular Triangle in Numbers. The like for others.

To find a Rectangular Triangle in Numbers, *the Product of any two Numbers is one Side, the* double of *Difference of their Squares is the other Side, and the Sum of the same Squares is the Hypotenuse.*

Thus by these two Numbers, 2, 3, which are call'd *Generating Numbers*, the double 12, of their Product 6, is the Side AE, the Difference of their Squares 4, 9, is the Side AC, and the Sum 13, of the same Squares 4, 9, is the Hypotenuse CE.

The SECOND BOOK of EUCLID's ELEMENTS.

EUclid after having explain'd in the preceding Book, the Properties of the Parallelogram in general, treats in this, particularly of Rectangular Parallelograms, which are call'd by one only Name, *Rectangles*: Comparing together the Squares and the Rectangles which are form'd by a Right-Line vairously cut, and of its Parts.

Altho' this Book seems difficult, yet it will prove very easy to him, who shall examine with Attention its Propositions, most of the Demonstrations whereof will be conceiv'd by regarding simply the Figure, being founded only upon this clear and evident Principle, which teaches us, that *the whole is equal to all its Parts taken together*.

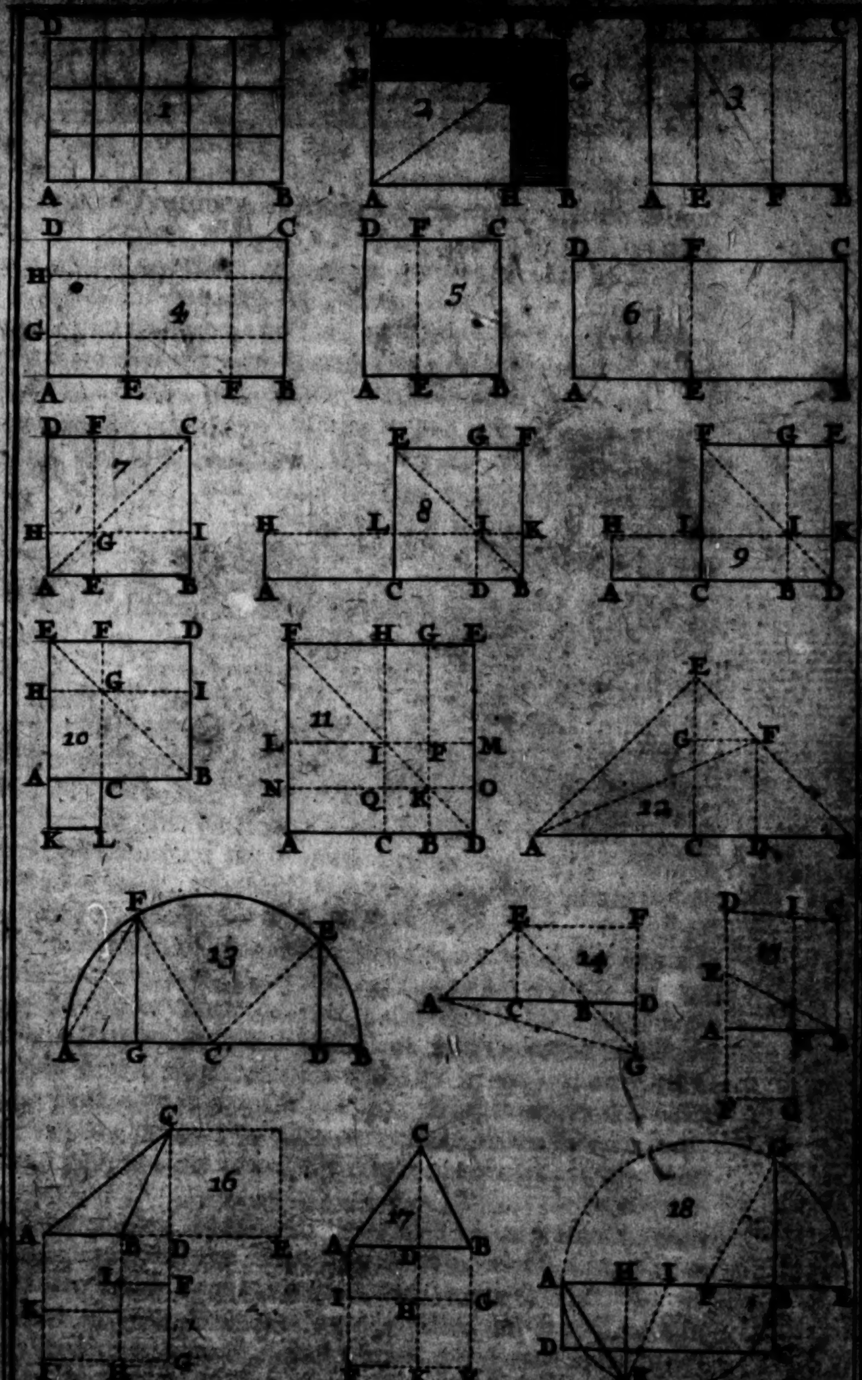
DEFINITIONS.

I.

Fig. 1. The *Rectangle contain'd under two Lines* is that where those two Lines, which represent the Length and the Breadth thereof form a Right-Angle. *Thus it is known, that the Rectangle ABCD, is compris'd under the two Lines AB, AD, which form the Right-Angle A, the Line AB representing the Length, and AD the Breadth.*

A Rectangle is seldom other than imaginary, because it suffices that the Length of it AB, and the Breadth AD is given, to conceive that of these two Lines AB, AD, it is possible to form a Rectangle thereof, which becomes a Square, when these two Lines are equal to each other.

A Quantity of the Surface of a Rectangle, that is to say, the Area of a Rectangle is measur'd by little Squares, as by square Feet, or by square Yards, according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cording as the Length and the Breadth are expreſs'd in Fig. 1. Feet or in Yards.

The Neceſſity of this Meaſure proceeds from a Surface being produc'd by the motion of a Line, which produces the Lines that compoſe the Surface, by the infinite Number of Points, whereof the Line which is mov'd is compos'd; as a Rectangle by the motion of a Line along another, which is perpendicular to it.

Thus if the Breadth AD, is compos'd, for example of three Points, that is to ſay, of three Feet, by taking a Foot for a Point; and if this Line AD, is mov'd along the Breadth AB, which we will ſuppoſe five Feet, by ſtill keeping at Right-Angles; it will deſcribe by its continual motion, Right-Lines, which will interſect at Right-Angles; and will make as many ſquare Feet as you ſee mark'd in the Figure, to wit 15, which may be found compendiouſly by multiplying the Length by the Breadth, that is to ſay, five by three.

This is the reaſon why the ſaid Rectangle is ſometimes expreſs'd in *Numbers*, without being actually deſcrib'd, to wit, by multiplying together the Numbers of the Meaſures of the two Lines which form it, to ſhew by the Product of the Multiplication, that the Rectangle, which is conceiv'd, made under theſe two Lines, hath as many ſuch like ſquare Meaſures in its Superficies; and 'tis for this Reaſon that the Number produc'd by the Multiplication of theſe two others, is call'd by *Euclid*, *a Plane Number*, whereof the two other Numbers which produce it, are call'd the *Sides*.

The Reaſon of this Multiplication is evident, becauſe if the Length AB, be but a Foot, the Line AD, in paſſing over that Foot of the Line AB, wou'd produce a Row of three ſquare Feet; but as the Length AB, is ſuppos'd five Feet, the Line AD, in going over thoſe five Feet, wou'd produce five Rows of three ſquare Feet each, that is to ſay, five times three ſquare Feet, or 15 ſquare Feet for the intire Superficies of the Rectangle ABCD.

Now as the Length AB, may alſo be imagined to move along the Breadth AD, to produce the ſame Plane ABCD, it is evident that the Length AB, by being mov'd one Foot, along the Line AD, will produce a Row of five ſquare Feet; and that in being mov'd three Feet, that is to ſay, in going over the whole Line AD, ſtill parallel to its ſelf, will produce three Rows of five ſquare Feet, that is to ſay, three times five ſquare Feet, or fifteen ſquare Feet as before, for the

Fig. 1. Surface ABCD. Where you see that *two Numbers being multiplyed reciprocally, the one by the other, produce one and the same Number*. As here by multiplying 3 by 5, the same Number is produced as by multiplying 5 by 3, to wit, 15.

II.

Fig 2. If through a Point E, taken at discretion, upon the Diagonal AC, of the Rectangle ABCD, you draw to the two Sides AB, AD, the two Parallels FG, HI, there will be form'd four little Rectangles, whereof the two DE, BE, through which the Diagonal passes not, with the one of the other two, as with GI, form the Figure BCF, which is call'd *Gnomon*, because it resembles a Carpenter's Square.

PROPOSITION I.

THEOREM I.

If of two Right-Lines, the one is cut in as many Parts as you will, the Rectangle compris'd under those two Lines is equal to the Rectangles compris'd under the Line which is not divided, and under the Parts of that which is divided.

Fig. 3. I Say, that if of the two Lines AB, AD, the first AB, be divided at the Points, E, F, the Rectangle ABCD, compris'd under those two Lines, is equal to all the Rectangles compris'd under the Line AD, which is not divided, and under the Parts AE, EF, BF, of the divided Line AB. So that if the Line AD, is for example 10 Feet, the Line AB 12. and its Parts AE, 3, EF, 5, and BF, 4. the Rectangle in Numbers under these two Lines 12, 10, to wit, 120, is equal to the Rectangle 30, under AD, AE, the Rectangle 50, under AD, EF, and the Rectangle 40, under AD, BF.

PREPARATION.

Draw from the Points of division E, F, the Right-Lines EG, FH, perpendicular to the Line AB, the which will be parallel to each other, and to the Sides AD, BC, as is evident *per* 28. 1. and *per* 30. 1. by reason of the four Right-Angles A, E, F, B, and more than that, they will be equal to each other *per* 34. 1. by reason of the three Parallelograms AG, EH, FC.

DE-

Fig. 3.

DEMONSTRATION.

Since the Rectangle AG, is made under the Line AD and the first Part AE, the Rectangle EH, is made under the Line EG, or AD, its equal, and the other Part EF, and the Rectangle FC, is made under the Line FH, or AD, its equal, and the last Part BF; and since these three Rectangles AG, EH, FC, agree with the Rectangle ABCD, to which *per Ax.* 8. they are equal, it follows that the Rectangle ABCD, is equal to the Sum of all the Rectangles compris'd under the Line AD, and each Part of the other Line AB. *Which was to be shewn.*

USE.

This Proposition serves for the Demonstration of the ordinary Practice of Multiplication, at least when you multiply a Number composed of several Figures, by another Number of a single Figure. For Example, when you wou'd multiply 312 by 3, you must take this Number 3 for the Line AD, and the first Number 312, for the Line AB, and its Parts 300 for AE, 10 for EF, and 2 for BF, the which being multiplyed separately by 3, you have 900 for the Rectangles AG, 30 for the Rectangle EH, 6 for the Rectangle FC, and the Sum 936, of these three Rectangles, give the Rectangle ABCD, for the Product of the Multiplication.

In like manner to multiply $a+b+c$ by d, you must take d for AD, and $a+b+c$ for AB, and its Parts a for AE, b for EF, and c for BF, the which being multiplied separately by d, produces ad, for the Rectangle AG, bd for the Rectangle EH, cd for the Rectangle FC, and the Sum $ad+bd+cd$ of those three Rectangles give the Area of the Rectangle ABCD, for the Product of the Multiplication.

The whole Practice of Multiplication, cannot be demonstrated by this Proposition nor the following ones, for when there is to be multiplyed together two Numbers compos'd each of several Figures, to demonstrate the ordinary Practice us'd in this Multiplication, there is need of a Theorem more general than the preceding, to wit, that *the Rectangle under two right Lines cut as you please, is equal to all the Rectangles made under the Parts of the one and the Parts of the other.* That is to say, if the Line AB, be cut at the Points E, F, and the Fig. 4. Line

Fig. 4. Line AD, at the Points G, H, the Rectangle ABCD, under those two Lines is equal to all the Rectangles compris'd under the Parts of the Line AB, and the Parts of the Line AD; as will be easily seen by drawing from the Points of Division, perpendiculars to each Line.

PROPOSITION. II.

THEOREM II.

The Square of a Line divided as you will, is equal to all the Rectangles compris'd under the whole Line, and each of its Parts.

Fig. 5. ALthough this Proposition be a Corollary of the Preceding, nevertheless we shall demonstrate it particularly, after *Euclid*'s manner.

I say then, that if the Line AB, be divided for example in two Parts at the Point E, its Square ABCD, is equal to all the Rectangles compris'd under the same Line AB, and each of its Parts. So that if the Part AE, is for example 3 Feet, and the Part EB 5, so that the whole Line AB or AD, be 8 Feet, in which Case the Square ABCD, will be 64 Feet square, because that 8 multiplyed by 8 makes 64, the which Number is equal to the Number 24 square Feet of the Rectangle AF, and to the Number 40 square Feet of the Rectangle EC.

PREPARATION.

Draw from the Point of Division E, the Right-Line EF, perpendicular to the Line AB, which will divide the Square ABCD, in two Rectangles AF, EC, whereof the Sides AD, EF, will be equal to the Line AB.

DEMONSTRATION.

Since the Rectangle AF, is made under the first Part AE, and the Line AD, equal to the Line AB; and since the Rectangle EC is compris'd under the other Part EB, and the Line EF, equal to the same Line AB; and that these two Rectangles AF, EC, agree with the

Square ABCD, it follows *per Ax.* 8. that the Square ABCD is equal to them. *Which was to be shewn.* Fig. 5.

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USE.

This Proposition serves for the Demonstration of *Prop.* 4. by a Method, which will serve for the second Demonstration to *Prop.* 2. to wit, by Analysis, thus,

If the Letter *a* be put for the Part AE, and the Letter *b* for the other Part EB, so that the whole Line AB, or AD, be $a+b$, the Rectangle AF, will be $aa+ab$, and the Rectangle EC, will be $ab+bb$, and the Sum of those two Rectangles will be $aa+2ab+bb$ for the Square ABCD, where you see that this Square is equal to the two Squares aa, bb, of the two Parts AE, EB, and to the double Rectangle $2ab$ under the same Parts, as *Prop.* 4. emports.

PROPOSITION III.

THEOREM III.

If you divide at pleasure a Line in two; the Rectangle compris'd under the whole Line, and one of its Parts, is equal to the Square of that Part, and to the Rectangle under the two Parts.

I Say, that if the Line AB, be divided as you will in E, the Rectangle ABCD, under that Line AB, and the Part AE, so that AD, AE, be two equal Lines; is equal to the Square of the same Part AE, and to the Rectangle under the two Parts AE, BE. Fig. 6.

PREPARATION.

Draw from the Point of Division E, the right Line EF, perpendicular to the Line AB, the which perpendicular will be equal to the Part AE, because it is parallel and equal to the Line AD, which is suppos'd equal to the Part AE; which makes that the Rectangle AF, is the Square of the Part AE, and EC the Rectangle under the two Parts AE, EB.

DEMONSTRATION.

Since the Rectangle AF is the Square of the Part AE; and

Fig. 6. and since the Rectangle EC is made under the two Parts AE, BE, and since those two Rectangles AF, EC, agree with the Rectangle ABCD, it follows *per Ax.* 8. that the Rectangle ABCD, is equal to the Square AF, of the Part AE, and to the Rectangle EC, under the Parts AE, BE. *Which was to be demonstrated.*

SCHOLIUM.

The Mind may be convinc'd of the Truth of this Theorem without any Preparation, to wit, by Analysis, by putting the Letter a for the Part AE, and the Letter b for the other Part BE, so that the whole Line AB, be $a+b$, the which being multiplyed by AD or AE, or a comes $aa+ab$ for the Rectangle ABCD, the which is equal as you see, to the Square aa of the Part AE, and to the Rectangle ab under the Parts AE, BE. *Which was to be shewn.*

USE.

This Proposition may serve for the Demonstration of the following, and also of *Prop.* 14. and is made use of upon several other occasions, for the ready and easy demonstration of more difficult Theorems.

PROPOSITION IV.

THEOREM IV.

The Square of a Line divided in two at pleasure, is equal to the Squares of its two Parts, and to two Rectangles under the same Parts.

Fig. 7. I Say, that the Square ABCD, of the Line AB, cut as you will at the Point E, is equal to the Squares of the Parts AE, BE, and to two Rectangles under the same Parts AE, BE. So that if the Part AE, is for example 3 Feet, and the Part BE, 6, so that the whole Line AB be 9 Feet, the Square ABCD, which will be 81 Feet square, because 9 multiplyed by 9 makes 81, is equal to the Square 9 of the Part AE, to the Square 36, of the other Part BE, and to the two Rectangles under the Parts AE, BE, that is to say, to twice 18 or to 36.

PRE-

Fig. 7.

PREPARATION.

Having drawn the Diagonal AC, draw from the Point E, the right EF perpendicular to the Line AB, and through the Point G, where it cuts the Diagonal AC, draw to the same Line AB, the Parallel HI, the which with the first EF, divides the Square ABCD, in four Rectangles, to wit, AG, BG, CG, DG.

DEMONSTRATION.

By reason of the two equal Sides BA, BC, of the Triangle ABC *per constr.* the two Angles BAC, ACB, will be equal to each other, *per* 5. 1. and each will be a semi-right one *per* 32. 1. because together they make a right one, by reason of the Angle B, which is a right one, since it is the Angle of a Square.

It will be known in the same manner that the two Angles DAC, DCA, of the Rectangular Isoscele Triangle ADC, are each a semi-right one. From whence it follows *per* 32. 1. that by reason of the right Angles E, H, I, the Angles AGE, AGH, CGF, CGI, are also semi-right ones, and consequently equal to each other, and *per* 6. 1. that the two Lines AE, GE, are equal to each other, as well as the two AH, GH, and as the two GI, CI, and again as the two CF, GF.

Because the opposite Sides of a Parallelogram are equal to each other, *per* 34. 1. it is easy to conclude that the Rectangle AG, is the Square of the Part AE, that the Rectangle FI, is the Square of the other Part BE, and that each of the two Rectangles BG, DG, is made under the same Parts AE, BE, and since these four Rectangles AG, FI, BG, DG, agree with the Square ABCD, it follows by *Ax.* 8. that they are equal to it. *Which was to be demonstrated.*

SCHOLIUM.

This Proposition may be demonstrated by means of the preceding, without the Diagonal AC, to wit, by making AH equal to the Part AE, and by drawing from the Point E the Line EF, perpendicular to the Line AB, and from the Point H the Line HI, perpendicular to the Line AD, and by reasoning in this manner.

The

Fig. 7. The Rectangle AI, under the Line AB, and the Part AE is equal to the Square AG of this Part AE, and to the Rectangle EI, under the Parts AE, BE, by *Prop.* 3. and likewise the Rectangle DI, under the same Line AB, and the other Part BE is equal to the Square FI, of this Part BE, and to the Rectangle DG, under the Parts AE, BE; but the two Rectangles AI, DI, are together equal to the Square ABCD, as you see: therefore the Squares of the two Parts AE, BE, with the double Rectangle under the same Parts AE, BE, are also together equal to the Square ABCD. *Which was to be demonstrated.*

The Analysis discovers and demonstrates also at the same time the Truth of this Theorem, for if you put the Letter a for the Part AE, and the Letter b for the other Part BE, so that the Line AB be $a+b$, by multiplying $a+b$ by its self, that is to say, by $a+b$, you have $aa+2ab+bb$ for the Area of the Square ABCD, where you see that this Area is equal to the Squares aa, bb, of the two Parts AE, BE, and to the double Rectangle $2ab$ under the same Parts AE, BE. *Which was to be demonstrated.*

USE.

This Proposition serves for the Demonstration of the following ones, and principally for the Demonstration of *Prop.* 12. It is the Foundation of the Method commonly us'd in finding the Square Root of a Number compos'd of more than two Figures. As if the Number be 529, you must consider this Number 529, as the Area of the Square ABCD, whereof the Side of the Square is sought in Numbers, which is that which is call'd *Square Root*, the which ought to have in this Example two Figures, which are represented by the Parts AE, BE.

When you take the square Root of 5, which is equivalent to 500, you have 2 or 20 for the bigger Part BE, whereof the Square is 4 or 400, which is represented by the Square FI, being taken away from 529, which represents the Square ABCD, there remains 129, for the Gnomon FAI, which comprehends the two equal Rectangles FH, BG, and the Square AG of the Part AE, which represents the second Figure of the Root which is sought.

To find this second Figure, it is conceiv'd that these two equal Rectangles FH, BG, are set in the right Line, to the end that together they should make a single Rectangle, whereof the Base will be 4 or 40, to wit, the double

double of the first Figure found; because this single Fig. 7.
Rectangle, with the Square AG, make a whole Rectangle, whi[illegible] equivalent to 129, if 129 be divided by the doub[illegible]o, you'll find 3 in the Quotient for the second Figure of the Root which is look'd for, the which consequently will be equivalent to 20+3, or 23; and when you have multiplyed the Divisor 40 by 3, and substracted the Product 120, which is the Sum of the two equal Rectangles DG, BG, there remains again 9, for the Square AG, so that from the Remainder 9, you ought to substract again the Square 9, of the second found Figure 3.

The indetermin'd Square $aa+2ab+bb$, whereof the Square Root $a+b$ is sufficient to find the Square Root of a Number, as of the same Number 529; for when from this Number 529, you substract the Square 400, of the first found Figure 20, which the Letter a represents, it is as if from $aa+2ab+bb$ you have substracted the Square aa, and then the remainder 129 will be represented by the rest $2ab+bb$, which shews that to find the second Figure, which the Letter b represents, you must divide the Remainder by the double of the first, by reason of $2ab$, &c.

COROLLARY I.

It follows from this Proposition, that the Diagonal of a Square divides each of the two opposite Angles equally in two, and that the Rectangles through which it passes, as EH, FI, are Squares.

COROLLARY II.

It follows also that of any two Numbers, the Sum of their Squares with the double of their Product makes one square Number, to wit the Square of the Sum of those two Numbers.

PROPOSITION V.

THEOREM V.

If a Right-Line is cut equally and unequally, the Rectangle compris'd under the unequal Parts, with the Square of the Part between the two Section Points, is equal to the Square of half the Line.

I Say, that if the Line AB, be cut equally in two at the Point C, and unequally in two at the Point D, so that the unequal Parts be AD, DB; the Rectangle compris'd Fig. 8.

Fig. 8. pris'd under those two unequal Parts AD, BD, with the Square of the Part CD, terminated by the two section Points C, D, is equal to the Square BCEF, of the half BC, of the Line AB.

That is to say, that if the Line AB is for example 12 Feet, and its half AC, or BC, consequently 6, the intercepted Part CD, 4, and consequently the great unequal Part AD 10, and the little Part BD, 2, the Rectangle 20, of those two unequal Parts 10, 2, with the Square 16 of the intercepted Part 4, is equal to the Square 36 of the half 6, of the Line AB.

PREPARATION.

Having drawn the Diagonal BE, draw from the Point D, the Line DG, perpendicular to the Line AB, and through the Point I, where it cuts the Diagonal BE, draw the Line KL, perpendicular to the Line DG, and those two Perpendiculars DG, KL, will divide the Square BCEF into four Rectangles, whereof the two CI, FI, will be equal to each other, by *Prop*. 4. and the two others DK, LG, will be Squares by the same *Prop*. Raise again from the Point A, upon AB, the perpendicular AH, which meeting the Line KL, extended, in the Point H, will be *per* 34. 1. equal to the Line BK, or to the unequal Part BD, insomuch that the Rectangle AI, is compris'd under the unequal Parts AD, BD.

DEMONSTRATION.

Because the two Rectangles AL, CK, are compris'd under equal Lines, they will be equal to each other, as well as the two CI, FI, the which being join'd to the two preceding, each to each, sheweth that the Rectangle AI, under the unequal Parts AD, BD, is equal to the Gnomon FBL; and because this Gnomon FBL, with the Square GL, of the intercepted Part CD, is equal to the Square BCEF, it follows that the Rectangle under the unequal Parts AD, BD, with the Square GL, of the intercepted Part CD, is also equal to the Square BCEF. *Which was to be demonstrated.*

Fig. 8.

SCHOLIUM.

You may dispense with the Square BCEF, and be contented with the Rectangle AK, compris'd under the Line AB, and its little unequal Part BD, equal to BK, or AH, and the two perpendiculars have CL, DI, to have this demonstrated.

Because the Square of the Line BC is equal by *Prop.* 4. to the Squares of the Lines CD, BD, and to the two Rectangles under the same Lines CD, BD, that is to say, to the double Rectangle CI, and that instead of a Rectangle CI, and of a Square of the Line BD, that is to say, of the Square DK, the single Rectangle CK, or CH, its equal may be put; it is plain that the Square of the Line BC, is equal to the Square of the Line CD, and to the two Rectangles CH, CI, that is to say to the single Right-Angle AI, under the unequal Parts AD, CD. *Which was to be demonstrated.*

This may also be very easily demonstrated by Analysis, thus,

If you put the Letter *a* for the half AC, or BC, and the Letter *b* for the intercepted Part CD, you will have $a+b$ for the greatest unequal Part AD, and $a-b$ for the least BD: and if you multiply those two Parts together AD, BD, or $a+b$, $a-b$, you will have $aa-bb$, for the Rectangle under the same Parts AD, BD, to which adding the Square bb of the intercepted Part CD, you will have aa for the Sum of the Rectangle under the unequal Parts AD, BD, and of the Square of the intercepted Part CD, the which Sum, as you see, is fully equal to the Square of the half BC. *Which was to be demonstrated.*

USE.

This Proposition serves to demonstrate *Prop.* 14. and also *Prop.* 35. 3. and to demonstrate the principal Properties of the Ellipsis, as may be seen in the Treatise that we have heretofore publish'd concerning *Lines of the second kind.*

It is the Foundation of all Quadratick Equations, or Equations of two Dimensions, and of the Method that is commonly us'd to find the Square Root of a Binomial, where one of the Terms is a Rational Number, and the Square of the other also a Rational Number.

This Proposition serves also to demonstrate, that *the Product under the Sum, and the Difference of two unequal Numbers, is equal to the Difference of their Squares*; being 'tis evident

Fig. 8. evident that AD is the Sum, and BD the Difference of two Numbers express'd by the Lines AC, CD, and that the Excess of the Square CF, of the greater Number BC, or AC, above the Square GL, of the lesser Number CD, to wit, the Gnomon FBL, is equal to the Rectangle under the Sum AD, and the Difference BD of the same two Numbers, AC, CD; besides that this Rectangle hath been found in Letters to be *aa — bb*, to wit, the Difference of the Squares of the Numbers AC, CD, because the Letter *a* hath been put for AC, and the Letter *b* for CD.

COROLLARY.

From whence it follows, that the *Difference of two Squares is divisible by the Sum or by the Difference of their Sides*: which serves to find by Calculation the Roots of Equations of two Dimensions, as we have taught towards the end of our *Treatise of Lines of the second kind*.

It follows also that, *if to the Product of two unequal Numbers, the Square of half their Difference be added, there will be produced a square Number*: to wit, the Square of half their Sum; it being certain that as AC, or BC, is half the Sum of the two Quantities AD, DB, so CD is half their Difference, because as the greater AD, surpasses the half AC, by CD, so the less BD, is surpass'd by the same half AC, or BC, by the same Quantity CD.

PROPOSITION VI.

THEOREM VI.

If a Right-Line be added to another divided equally in two, the Rectangle compris'd under the whole Line, and under the added one, with the Square of half the divided Line, is equal to the Square of a Line compos'd of the added one, and of half the divided one.

Fig. 9. I Say, that if to the Line AB, which is divided equally in two at the Point C, the Line BD be added to it, of what bigness you will, the Rectangle under the whole Line AD, and under the added one BD, with the Square of the half AC, or BC, is equal to the Square CDEF, of the Line CD, compos'd of the half BC, and of that added BD.

That is to say, that if the Line AB, is for example 10 Feet, and the added one BD, 2, and consequently the half AC, or BC, 5. the compos'd one CD, 7, and the whole

whole one AD, 12; the Rectangle 24, under the Line AD and the Line BD, with the Square 25, of the half BC, is equal to the Square 49, of the Line CD, which is 7 Feet. Fig. 2.

PREPARATION.

Having drawn the Diagonal DF, raise from the Point B, the Line BG, perpendicular to the Line AD, and through the Point I, where it cuts the Diagonal DF, draw the Line KL, perpendicular to the Line BG, and these two Perpendiculars BG, KL, will divide the Square CDEF, into four Rectangles CI, DI, EI, FI, whereof the two DI, FI, are Squares by *Prop.* 4. and the two others CI, EI, are equal to each other, *by the same.* Again, from the Point A, erect AH perpendicular to AB, which will meet the Line KL, prolong'd at the Point H, and will make the Rectangle AL, equal to the Rectangle CI, and consequently to the Rectangle EI, since these Rectangles have the same Length and the same Breadth.

DEMONSTRATION.

If to each of the two equal Rectangles AL, EI, the common Rectangle CK be added, you will have the Rectangle AK, equal to the Gnomon EDL, and if to each of these two equal Quantities, the common Square GL be added, you will find that the Rectangle AK, together with the Square GL, that is to say, the Rectangle under the Lines AD, BD, together with the Square of the half BC, is equal to the Square CDEF. *Which was to be demonstrated.*

SCHOLIUM.

This Proposition may also be demonstrated very easily by the new Analysis, by putting the Letter a for the half AC, or BC, and the Letter b for the added Line BD, and then will be had $2a$, for the Line AB, $a+b$, for the Line CD, and $2a+b$ for the Line AD, and the Rectangle under AD and BD will be $2ab+bb$, to which adding the Square aa of the half BC, you'll have $aa+2ab+bb$ for the Sum of the Rectangle under the Lines AD, BD, and of the Square of the half BC, which Sum $aa+2ab+bb$ is, as you see, equal to the Square of the Line CD, which is equivalent to $a+b$, because multiplying $a+b$ by

Fig. 9. by $a+b$ there comes $aa+2ab+bb$. *Which was to be demonstrated.*

USE.

This Proposition serves to demonstrate *Prop.* 11. and also *Prop.* 36. 3, and to demonstrate the principal Properties of the *Hyperbola*, as may be seen in the *Treatise of Lines of the second kind*, which we have publish'd heretofore: It serves also to resolve Equations of two Dimensions, and upon several other Occasions.

COROLLARY.

It follows also from this Proposition, that *if to the Product of two unequal Numbers, you add the Square of half of their Difference, the Sum will be a Square Number*, to wit, the Square of half the Sum of those two Numbers; it being certain that as AC, or BC, is half the Difference of the two Quantities AD, BD, which represents the two Numbers, so CD is half their Sum, as will be known by adding to the greater Number AD, the least BD, in a Right-Line towards A, to have their Sum, whereof CD will be the half.

PROPOSITION VII.

THEOREM VII.

The Square of a Line divided into two Parts at pleasure, with that of the one of its two Parts, are together equal to two Rectangles under that Line, and the same Part, and to the Square of the other Part.

Fig. 10. I Say, that the Square ABDE, of the Line AB, cut at pleasure, as suppose in the Point C, with the Square ACLK, of its Part AC, are together equal to two Rectangles compris'd under the Line AB, and the same Part AC, and to the Square of the other Part BC.

That is to say, that if the Line AB, is for example 12 Feet, its Part AC, 5, and consequently the other Part BC, 7, the Square 144 of the Line AB, with the Square 25, of the Part AC, makes the Sum 169, equal to 120, which is the double Rectangle under the Line AB, and the same Part AC, and to the Square 49, of the other Part BC.

PRE-

PREPARATION.

Fig. 10.

Having drawn the Diagonal BE, prolong the Line CL to F, and through the Point G, where the Line CF cuts the Diagonal BE, draw the Line HI, perpendicular to the Line CF, and these two Perpendiculars CF, HI, will divide the Square ABDE, into four Rectangles, whereof the two CI, FH, are two Squares, and the two others AG, DG, are equal to each other, by *Prop.* 4.

DEMONSTRATION.

If to the two equal Rectangles AG, DG, the two equal Squares AL, FH, be added, the two equal Rectangles GK, DH, will be had, whereof each is compris'd under the Line AB, and its Part AC, so that the Sum of those two equal Rectangles, that is to say, the Figure DHL is equal to two Rectangles under the Line AB, and its Part AC; wherefore if to each of these two equal Quantities you add the Square CI, then will the Figure DHL, with the Square CI, that is to say, the Square AD of the Line AB, with the Square AL, of its Part AC, are together equal to two Rectangles under the Line AB, and the same Part AC, and to the Square of the other Part BC. *Which was to be demonstrated.*

SCHOLIUM.

This Theorem may be demonstrated by the new Analysis, by putting the Letter a for the Part AC, and the Letter b for the other Part BC, and then you will have $a+b$ for the Line AB, and $aa+ab$ for the Rectangle under the Line AB, and its Part AC, and the double of this Rectangle will be $2aa+2ab$, to which adding the Square bb of the other Part BC, you will have $2aa+2ab+bb$, for the Sum of the two Rectangles under the Line AB, and its Part AC, and of the Square of the other Part BC, the which Sum $2aa+2ab+bb$, is equal to the Sum of the Square $aa+2ab+bb$, of the Line AB, and of the Square aa of the first Part AC. *Which was to be demonstrated.*

USE.

Fig. 10.

This Proposition doth not seem of any great Use in the Mathematicks, and it seems as if *Euclid* put it here only as a Lemma to *Prop.* 13.

PROPOSITION VIII.

THEOREM VIII.

If a Line cut in some Point at pleasure is propos'd, and one of its Parts be added to it, the Square of the whole Line is equal to four Rectangles under the propos'd Line and under that Part, and to the Square of the other Part.

Fig. 11. I Say, that if the Line AB be cut in C, as you please, and you add to it the Line BD, equal to the Part, BC; the Square ADEF, of the whole AD, is equal to four Rectangles under the Line AB, and its Part BC, or BD, and to the Square of the other Part AC.

That is to say, that if the Line AB, is for example 7 Feet, its Part AC 5, and consequently the other Part BC or BD 2, and the whole AD 9; the Square 81, of this Line AD, is equal to the Quadruple of the Rectangle 14, under the Line AB, and the Part BC, or BD, to wit to 56, and to the Square 25 of the other Part AC.

PREPARATION.

Having drawn the Diagonal DF, raise from the two Points B, C, the Lines BG, CH, perpendicular to the Line AB, and through the Points I, K, where they cut the Diagonal DF, draw the Lines LM, NO, parallel to the Line AB; and the Square ADEF, will be found divided into several Rectangles, among which the six LH, NG, PQ, PO, BQ, BO, will be Squares, whereof the four last PQ, PO, BQ, BO, will be equal to each other, because their Side are equal each to the Line BC, or BD.

DE-

DEMONSTRATION.

The Rectangles AK, NP, EK, are equal to each other, because they have one and the same Length equal to the Line AB, and one and the same Breadth equal to the Part BC, or BD: and the Rectangle GI, with the little Square BO, make likewise together a Rectangle equal to one of the three preceding, because they are equivalent to the single Rectangle GQ, by reason of the Square PO, equal to the Square BO. Thus you find precisely in the Square ADEF, four Rectangles under the Line AB, and its Part BC, or BD, and more than that, the Square LH, of the other Part AC. *Which was to be demonstrated.*

SCHOLIUM.

To demonstrate this Proposition by the new Analysis, put as usual, the Letter a for the Part AC, and the Letter b for the other Part BC, or BD, and then you have $a+b$ for the Line AB, $2b$ for the Line CD, and $a+2b$ for the whole Line AD, whose Square $aa+4ab+4bb$ is compos'd of the Quadruple $4ab+4bb$, of the Rectangle $ab+bb$ of the Line AB, and of the Part BC, or BD, and of the Square aa of the other Part AC, *Which was to be demonstrated.*

USE.

This Proposition serves to make out several Demonstrations in Geometry, and I have made very good use of it in my *Treatise of Lines of the second kind*, to demonstrate that the Focus of the *Parabola* is distant from the Vertex of the *Parabola*, by a Quantity equal to the fourth Part of the *Parameter*.

COROLLARY I.

It follows from this Proposition, *That if to quadruple the Product of any two Numbers, the Square of their Difference be added, the Sum will be a square Number*; to wit the

Fig. 11. Square of the Sum of those two Numbers, it being certain that the Line AD, is the Sum of the two Numbers represented by the Lines AB, BD, and that AC is their Difference, by reason of BC equal to BD.

COROLLARY II.

It follows also that *a Square is quadruple to another Square, when its Side is double the Side of that other Square*: it being evident that the Square CM, whereof the Side CD is double the Side BD, of the little Square BO, is quadruple that Square BO, because it comprehends four equal to it.

PROPOSITION IX.

THEOREM IX.

If a Line be cut equally and unequally, the Squares of the unequal Parts, will be together double the Sum of the Square of half the divided Line, and of the Square of the Part terminated by the two Points of Division.

Fig. 12. I Say, that if the Line AB be divided equally in the Point C, and unequally in the Point D, so that the two unequal Parts be AD, BD; the Squares of those two unequal Parts AD, BD, are together double the Squares of the Lines AC, CD, taken together.

That is to say, that if the Line AB, is for example 10 Feet, the intercepted Part CD, 2, and consequently the half AC, or BC, 5, the greatest unequal Part, AD, 7, and the less BD, 3; the Sum 58 of the Squares 49, 9, of the unequal Parts AD, BD, is double the Sum 29, of the Squares 25, 4, of the Lines, AC, CD.

PREPARATION.

Raise from the middle Point C, the right Line CE, perpendicular to the Line AB, and equal to its half AC, or BC, and join the Right-Lines AE, BE. Draw from the Point D, the Line DF, parallel to the Line CE, and from the Point F, the Right-Line FG, parallel to the Line

Line CD, and you'll have the Parallelogram CDFG, whereof the two opposite Sides CD, FG, will be equal to to each other by 34. 1. Lastly, join the Right-Line AF. Fig. 12.

DEMONSTRATION.

It will be known as in *Prop*. 4. that each of the acute Angles of the two Rectangular Isosceles Triangles ECA, ECB, is a semi-right one; and consequently the whole Angle AEB, is a right one. It appears also by 29. 1. and by 32. 1. that the two acute Angles of each of the two Rectangular Triangles EGF, FDB, is a semi-right one, and that by 6. 1. those two Triangles are Isosceles, that is to say, that the Line EG, is equal to the Line GF, or CD, its equal, and the Line DF to the Line DB.

Because by 47. 1. the Square of the Line AE, is equal to the Sum of the Squares of the two Lines AC, CE, which are equal to each other by *constr.* it follows that the Square of the Line AE, is double the Square AC, that is to say the Square of the Line AC; and thus it is we shall discourse hereafter. It appears likewise that the Square EF, is double the Square GF, or CD. From whence it follows that the Sum of the Squares AE, EF, or by 47. 1. the single Square AF, or again the Sum of the two AD, DF, or the two AD, DB, is double the Sum of the two AC, CD. *Which was to be demonstrated.*

SCHOLIUM.

To demonstrate this Theorem by the new Analysis, put the Letter a for the half AC, or BC, and the Letter b for the intercepted Part CD, the which being added to, and taken from the half AC, or BC, you will have $a+b$ for the greatest Part AD, whereof the Square is $aa+2ab+bb$, and $a-b$ for the least Part BD, whereof the Square $aa-2ab+bb$ being added to the preceding Square $aa+2ab+bb$ of the greatest Part AD, you will have $2aa+2bb$ for the Sum of the two Squares AD, BD, the which is double, as you see to the Sum $aa+bb$, of the Square aa of the half AC, and of the Square bb of the intercepted Part CD. *Which was to be demonstrated.*

Fig. 13.

USE.

This Proposition serves to demonstrate that the *Squares of the versed Sine of an Angle of 45 Degrees, of the versed Sine of an Angle, which is the remainder of the precedent from a Semi-circle, that is to say, 135 Degrees, are together triple the Square of the Radius.* That is to say, if in the Semi-circle ABE, the Centre whereof is C, and the Diameter is AB, the Arch EB is 45 Degrees, and that from the Point E, you draw the right Line ED, perpendicular to the Diameter AB; the Squares of the Lines AD, BD, which are the versed Sines of the Arches AE, BE, or of the Angles ACE, BCE, are together triple the Square of the Radius AC.

DEMONSTRATION.

Since the Angle ECD, of the Rectangular Triangle CDE, is a semi-right one, by *Sup.* the Angle CED, will be also a semi-right one, by 32. 1. and by 6. 1. the Lines CD, DE, will be equal to each other, and the Square of the Radius CE, or AC, being by 47. 1. equal to the Squares of the two equal Lines CD, DE, will be double the Square of each. Thus instead of double the Square CD, you may take the Square of the Radius AC.

Because by *Prop.* 9. the Squares of the Lines AD, BD, are together double the Square of the Radius AC, and the Square of the intercepted Part CD, if in the Place of double the Square of this intercepted Part CD, you take the Square of the Radius AC, which has been demonstrated equal to it, it will appear that the Squares of the Lines AD, BD, are together triple the Square AC. *Which was to be demonstrated.*

PROPOSITION X.

THEOREM X.

If one Right-Line be added to another equally divided, the Square of the Line compos'd of the two, with the Square of the added one, are together double the Square of half the divided Line, and the Square of the Line compos'd of this half, and of the added one.

Fig. 14.

I Say, that if the Line BD be added to the Line AB, divided equally in two at the Point C, the Square of the whole Line AD, with the Square of the Line added BD, are together double the Square of the half AC or BC,

BC, and of the Square of the Line CD, compos'd of the half BC, and of the added one BD. Fig. 14.

That is to say, if the Line AB, is for example 10 Feet, and the added Line BD, 3, in which Case the half AC, or BC will be 5, the Line CD 8, and the whole Line AD 13; the Sum 178, of the Square 169, of the whole Line AD, and of the Square 9, of the added Line BD, will be double the Sum of the Square 25, of the half AC, or BC, and of the Squre 64, of the Line CD, compos'd of the half BC, and of the added Line BD.

PREPARATION.

Raise from the Point C the Line CE, perpendicular to the Line AB, and equal to the half AC or BC, and join the Right-Lines AE, BE. Draw from the Point D, the Line DF, parallel to the Line CE, and from the Point E the Line EF, parallel to the Line CD, and you'll have the Parallelogram CEFD, whereof the two opposite Sides CD, EF, will be equal to each other, by 34. 1. Lastly, prolong the two Lines BE, DF, until they meet at the Point G, and join the Right-Line AG.

DEMONSTRATION.

It will appear as in *Prop.* 9. that the Angle AEG, is a right one, and it will not be difficult to discover that the two Rectangular Triangles BDG, EFG, are Isosceles, that is to say, that the Line DG is equal to the Line BD, and the Line FG equal to the Line EF, and consequently to the Line CD.

It will appear likewise, as in *Prop.* 9. that the Square AE, is double the Square AC, and the Square EG double the Square EF, or CD. From whence it follows that the Sum of the two Squares AE, EG, or by 47. 1. the single Square AG, or the sum of the two AD, DG, or of the two AD, BD, is double the sum of the two AC, CD. *Which was to be demonstrated.*

SCHOLIUM.

To demonstrate this Proposition by the new Analysis, put the Letter a for the half AC, or BC, and the Letter b for the added Line BD; in which Case you will have $2a$ for AB, $a+b$ for CD, and $2a+b$ for the whole Line

Fig. 14. AD, whose Square $4aa+4ab+bb$ being added the Square bb of the added Line BD, the Sum $4aa+4ab+2bb$ is, as you see, double the Sum $2aa+2ab+bb$ of the Square aa of the half AC, and of the Square $aa+2ab+bb$ of the Line CD, compos'd of the half and of the added Line. *Which was to be prov'd.*

USE.

This Proposition may serve to demonstrate that, *the Sum of the Squares of the versed Sine of an Angle of 60 Degrees, and of the versed Sine of an Angle which is the Remainder of the preceding to a Semi-circle, that is to say, 120 Degrees, is to the Square of the Radius, as 5 to 2.* That is to say, in the Semi-circle ABEF, the Centre whereof is C, and the Diameter is AB, the Arch AF is 60 Degrees, and that from the Point F, you draw the right FG, perpendicular to the Diameter AB; the Sum of the Squares of the Lines AG, BG, which are the versed Sines of the Arches AF, BF, or of the Angles ACF, BCF, is to the Square of the Radius BC, as 5 to 2, or the Square of the Radius BC, is to the Sum of the Squares of the versed Sines AG, BG, as 2 to 5.

Fig. 15.

DEMONSTRATION.

Because the Point C is the Centre of the Semi-circle ABE, the two Sides CA, CF, of the Triangle ACF, are equal to each other; and the Angles CAF, AFC, will be likewise equal to each other, by 5. 1. and because the Angle ACF is 60 Degrees by *Sup.* the two others CAF, AFC, will be together 120 Degrees by 32. 1. and consequently each will be 60 Degrees, because the half of 120 is 60. Thus the three Angles of the Triangle AFC, will be equal to each other, from whence it follows by *Prop. 6.* that this Triangle is equilateral, and consequently the Perpendicular FG divides the Base AC equally in two, because the two Rectangular Triangles AGF, CGF are equal to each other, by 26. 1.

Because the Line AC is divided equally in two at the Point G, and that the Line BC is added to it, it follows by *Prop.* 10. that the sum of the Squares of the whole AB, and of the Line added BC, is double the sum of the Squares AG, BG; and as the line AB is double the line BC, the Square AB will be quadruple the Square BC, by *Coroll. Prop.* 8. and the sum of the same Squares AB, BC will consequently be quintuple the Square BC. From whence

whence it may easily be concluded, that the quintuple of the Square of the Radius BC is double the Sum of the Squares of the versed Sines AG, BG, and that consequently the Square of the Radius BC, is to the Sum of the Squares of the versed Sines AG, BG, as 2 is to 5. *Which was to be demonstrated.* Fig. 13.

PROPOSITION XI.

PROBLEM I.

To cut a given Right-Line in two such Parts, that the Rectangle under the whole and one of its Parts, be equal to the Square of the other Part.

TO divide the given Line AB in the Point H, for Example, so that the Rectangle under the Line AB, and its Part BH, be equal to the Square of the other Part AH; describe by *Prop.* 46. 1. upon the Line AB the Square ABCD, and having divided the Side AD equally in two at the Point E, set the Length of the Line EB, upon the prolong'd Line AD, from E to F, upon the Line AF, describe the Square AFGH, which will give the Point H required. So that if the Line GH be extended to I, the Rectangle BI will be equal to the Square AG. Fig. 15.

DEMONSTRATION.

Because the Line AD, is divided equally in two at the Point E, by *const.* and that the Line AF is added to it, it is plain by *Prop.* 7. that the Rectangle under the whole DF and the Line added AF, that is to say, the Rectangle DG, with the Square of the half AE, is equal to the Square EF or EB, that is to say by 47. 1. to the two Squares AE, AB, taken together; wherefore if you take away from each Side the Square AE, there will remain the single Rectangle DG equal to the single Square ABCD; and if from these two equal Planes you substract the common Rectangle AI, it will appear that the Square AG is equal to the Rectangle BI. *Which was to be done and demonstrated.*

SCHOLIUM.

This Line AB, thus divided in H, is said by *Euclid*, to be *cut in mean and extream Proportion*; and the Part Def. 3. 6.

BH

Def. 3. 6. BH is less than the other Part AH; because it is less than AE, half of AB, by Reason of AB less by *Prop.* 19. 1. than EB, or than EF, and by substracting from those unequal Quantities AB, EF, the equal ones AH, AF, there remains BH, less than AE.

USE.

Fig. 15. Among the different Uses of this Line thus cut, we will only say in this Place that it serves to inscribe in a Circle a Regular *Pentagon*, and also a regular *Pentedecagon*, that is to say, a regular *Polygon* of fifteen Sides, as will be taught in *Prop.* 11. and 16. *of Book* 4.

It is likewise very successfully us'd to find the Sines of an Arch of 18 Degrees, because we shall shew in *Prop.* 10. 4. that the greater Part AH, is the Side of a regular *Decagon* inscribable in a Circle, whose Radius is AB, and consequently is the Chord of an Arch of 36 Degrees, whose half is the Sine of 18 Degrees. But to find this Chord AH, suppose the whole Sine AB, to be 100000 Parts, and consequently its half AE, will be 50000, add together the Square 10000000000, 2500000000 of those two Lines, and the Sum 12500000000, will be by 47. 1. the Square BE, wherefore by taking the Square Root of this Sum, you will have 111805, for the Line BE, or EF its equal, from whence substracting the Line AE, which is equivalent to 50000, you will have 61803 for AF, or for the Chord AH of 36 Degrees, whose half 30901 is the Sine of 18 Degrees.

PROPOSITION XII.

THEOREM XI.

Sa na *In obtuse-angled Triangles, the Square of the Side, opposite to the obtuse Angle, is equal to the Sum of the Squares of the two other Sides, and to two Rectangles equal to each other, whereof each is compris'd under one of the two Sides of the obtuse Angle, and the Part of that produc'd Side, intercepted between the obtuse Angle and the perpendicular drawn from the opposite Angle upon the same Side.*

Fig. 16. I Say, if from the acute Angle C, of the *Amblygon* or obtuse-angled Triangle ABC, you let fall upon its produc'd opposite Side AB, the Perpendicular CD, the Square of the Side AC, opposite to the obtuse Angle B, is equal to the two Squares AB, BC, and to two Rectangles

angles equal to each other, each of which is compris'd under the Side AB, and the Part BD, terminated by the obtuse Angle B, and by the Perpendicular CD. Fig. 16.

That is to say, if the Side AB, is for example 4 Feet, the Side BC 13, the Side AC, 15, and the Part BD, 5, in which case the Perpendicular CD will be 12 Feet; the Square 225 of the Side AC is equal to the Sum of the Square 16, of the Side AB, of the Square 169 of the Side BC, and of 40 the double of the Rectangle 20, under the Side AB, and the Part BD.

DEMONSTRATION.

Forasmuch as by *Prop.* 4. the Square AD is equal to the Squares AB, BD, and to two Rectangles under AB, BD, if to these two equal Quantities you add the Square CD, it will appear that the Sum of the two Squares AB, CD, or by 47. 1. the single Square AC, is equal to the Square AB, to the Sum of the two Squares BD, CD, that is to say, by 47. 1. to the Square BC, and to two Rectangles under AB, BD. *Which was to be demonstrated.*

SCHOLIUM.

To render the Demonstration of this Theorem plainer, make upon CD, the Square CE, upon AD, the Square AG, upon BD, the Square BF, and upon AB, the Square BK, and produce the Side BL, as far as H: and then it will appear that each of the two Rectangles HK, HF, is made under AB, BD, and those together with the Square BK, and the two Squares BF, CE, that is to say, by 47. 1. the Square BC, are equal to the two Squares AG, CE, or by 47. 1. to the single Square AC.

USE.

This Proposition serves to discover when there is an obtuse Angle in a Triangle, whose three Sides are known, to wit, when the Square of the Side opposite to that Angle shall be greater than the Sum of the Squares of the two other Sides.

It is us'd also to discover the Quantity of the Perpendicular of an obtuse angled Triangle, when it falls without, which always happens when it falls from one of the acute Angles, as we have shewn in *Prop.* 17. This Perpendicular, as CD, will be found by the means of the three known Sides of the Triangle ABC. Thus,

Because

Fig. 16. Because we have suppos'd the Side AB 4 Feet, the Side BC 13, and the Side AC 15, the Square of AC will be 225, the Square of AB will be 16, and the Square of BC, will be 169, the Sum of these two last, 16, 169, will be 185, the which being substracted from the first 225, there will remain 40, whose half 20, will be the Rectangle under AB, BD: wherefore if you divide this Rectangle 20, by its Breadth AB, which is suppos'd 4 Feet, you will have 5 Feet, for its Length BD, whose Square 25, being taken from the Square 169, of the Side BC, there will remain 144, for the Square of the Perpendicular CD, by 47. 1. wherefore if you take the Square Root of this remainder, 144, you will have 12 Feet, for the Perpendicular CD.

PROPOSITION XIII.

THEOREM XII.

In any Rectilinear Triangle whatsoever, the Square of the Side opposite to an acute Angle, with two Rectangles compris'd under the Side upon which falls the perpendicular from the opposite Angle, and under the Part compris'd between the perpendicular and the acute Angle, is equal to the Sum of the Squares of the two other Sides.

Fig. 17. I Say, if in the Triangle ABC, the Angle B is acute, the Square of the Side AC, opposite to that acute Angle B, with two Rectangles compris'd under the Side AB, and the Part BD compris'd between the acute Angle B, and the Perpendicular CD, which falls from the Angle C, opposite to the Side AB, is equal to the Sum of the Squares of the two other Sides AB, BC.

That is to say, if the Side AB, is for example 14 Feet, the Side BC, 13, the Side AC, 15, and the Part BD, 5, in which case the Perpendicular CD will be 12 Feet, the Sum 365 of the Square 225 of the Side AC, and 140 the double of the Rectangle 70, under AB and BD, is equal to the Sum of the Square 196 of the Side AB, and of the Square 169 of the Side BC.

DEMONSRATION.

Because that by *Prop.* 7. the Sum of the two Squares AB, BD, is equal to the Sum of the Square AD, and to the double Rectangles under AB, BD, if you add to each Side

Side the Square of the Perpendicular CD, it will appear Fig. 17. that the Sum of the Square AB, and of the two Squares BD, CD, that is to say, by 47. 1. of the Square BC, is equal to the Sum of the two Squares AD, CD, or by 47. 1. of the single Square AC, and of the double Rectangle under AB, BD. *Which was to be demonstrated.*

SCHOLIUM.

To render the Demonstration of this Theorem plainer, describe upon AB the Square AE, upon BD the Square DG, and produce the Side GH as far as I, and the Perpendicular CD as far as K; and then 'twill appear that each of the two Rectangles DE, AG, is made under the Lines AB, BD, and that the Rectangle IK is the Square of the Line AD. We shall take then the Square AD, for IK, and the double of the Rectangle under the Lines AB, BD, for the Sum of the two DE, AG; and as this Sum, with the Square IK, is equal to the Square AE, and to the Square DG, because in the Sum of the two Rectangles DE, AG, the Square DG is taken twice, if to each Side you add the Square CD, it will appear that the Sum of the double Rectangle under AB, BD, and of the two Squares AD, CD, that is to say, by 47. 1. of the single Square AC, is equal to the Sum of the Square AB, and of the two Squares BD, CD, or by 47. 1. of the single Square BC.

USE.

This Proposition serves to discover when a propos'd Angle is acute in a Triangle, whose three Sides is known, which will happen when the Square of the Side opposite to that Angle, is less than the Sum of the Squares of the two other Sides.

It is used also to find the Length of the Perpendicular of a Triangle, when it falls within, which will always happen, when each of the two Angles of the Base shall be acute. This Perpendicular, as CD, will be found by means of the three known Sides of the Triangle ABC, thus,

Because we have suppos'd the Side AB 14 Feet, the Side BC 13, and the Side AC 15, the Square AB 196, the Square BC will be 169, and the Square AC will be 225, the which being substracted from the Sum 365 of the two first 196, 169; there will remain 140, whose half 70, is the Rectangle under AB, BD; wherefore if you divide

divide 70 by 14, which is AB, you will have 5, for BD, the Square whereof 25, being subſtracted from the Square 169 of the Side BC, the remainder 144 will be the Square of the Perpendicular CD, by 47. 1. Wherefore the ſquare Root 12 of this Remainder 144, will be the Quantity of the Perpendicular CD.

PROPOSITION XIV.

PROBLEM II.

Fig. 18.

To reduce a Right-lin'd Figure given into a Square.

AS a Right-lin'd Figure may be reduc'd into a Rectangle by *Prop.* 45. 1. it is evident that to reduce a Right-lin'd Figure propoſed into a Square, you need only know how to reduce a given Rectangle into a Square, as ABCD, thus,

Having produc'd one of the Sides, as AB to E, ſo that the Line BE be equal to the other Side BC, and having divided the whole Line AE into two equal Parts in the Point F, deſcribe from this Point F, through the two Points A, E, the Semi-circle AGE, and produce the Side BC, as far as G. The Line BG will be the Side of a Square equal to the propos'd Rectangle ABCD.

DEMONSTRATION.

Foraſmuch as the Line AE is cut in two equal Parts in the Point F, and into two unequal Parts in the Point B, the Rectangle under the unequal Parts AB, BF, that is to ſay, AC, with the Square of the intercepted Part FB, is by *Prop.* 5. equal to the Square FE, or FG, that is to ſay, by 47. 1. to the two Squares BF, BG, wherefore taking away the common Square BF, there remains the Rectangle AC, equal to the Square BG. *Which was to be done and demonſtrated.*

SCHOLIUM.

Without producing the Side AB, divide it into two equal Parts in the Point I, and deſcribe from this Point I, through the Points A, B, the Semi-circle AKB, and having taken the Line AH equal to the Side AD, draw from the Point H, the right HK, perpendicular to the Side AB, and through the Point K, where the Circumference AKB is cut by the Perpendicular HK, draw to the

the Point A, the Right-Line AK, whose Square will be equal to the Rectangle ABCD. Fig. 18.

DEMONSTRATION.

Because the Line AB, is cut into two equal Parts in the Point I, and into two unequal Parts in the Point H, the Rectangle under the unequal Parts AH, BH, with the Square of the intercepted Part HI, will be by *Prop.* 5. equal to the Square of the half AI, or IK, that is to say, by 47. 1. to the two Squares HK, HI; wherefore if you take away from each Side the Square HI, there will remain the single Rectangle under the Lines AH, BH, equal to the single Square HK, and if to each of these two equal Planes you add the Square AH, it will appear that the sum of the Rectangle under the Parts AH, BH, and of the Square AH, that is to say by *Prop.* 3, the propos'd Rectangle ABCD, is equal to the sum of the two Squares AH, HK, or by 47. 1. to the single Square AK. *Which was to be done and demonstrated.*

The Point H may happen to coincide with the Point I, to wit, when the Length AB shall be double the Breadth AD, in which case the Line HI, will be equal to 0, which alters the Demonstration so very little, that it is unnecessary to say more of it.

USE.

This Proposition serves for the Resolution of *Prop.* 25. 6. where this Problem is found resolv'd more generally.

The

The THIRD BOOK of

EUCLID's ELEMENTS.

EUclid explains in this Book the Nature and Properties of the most perfect Figure of all, which is the Circle, by comparing the several Lines which may be drawn as well within as without its Circumference, by the different Angles which are form'd there, and by the Contacts of a Right-Line, and of the Circumference of a Circle, or of two Circumferences of Circles: and he gives the first Principles of the Instruments which are used in *Astronomy*, and in other Arts, which are hardly to be done without the Circle.

DEFINITIONS.

I.

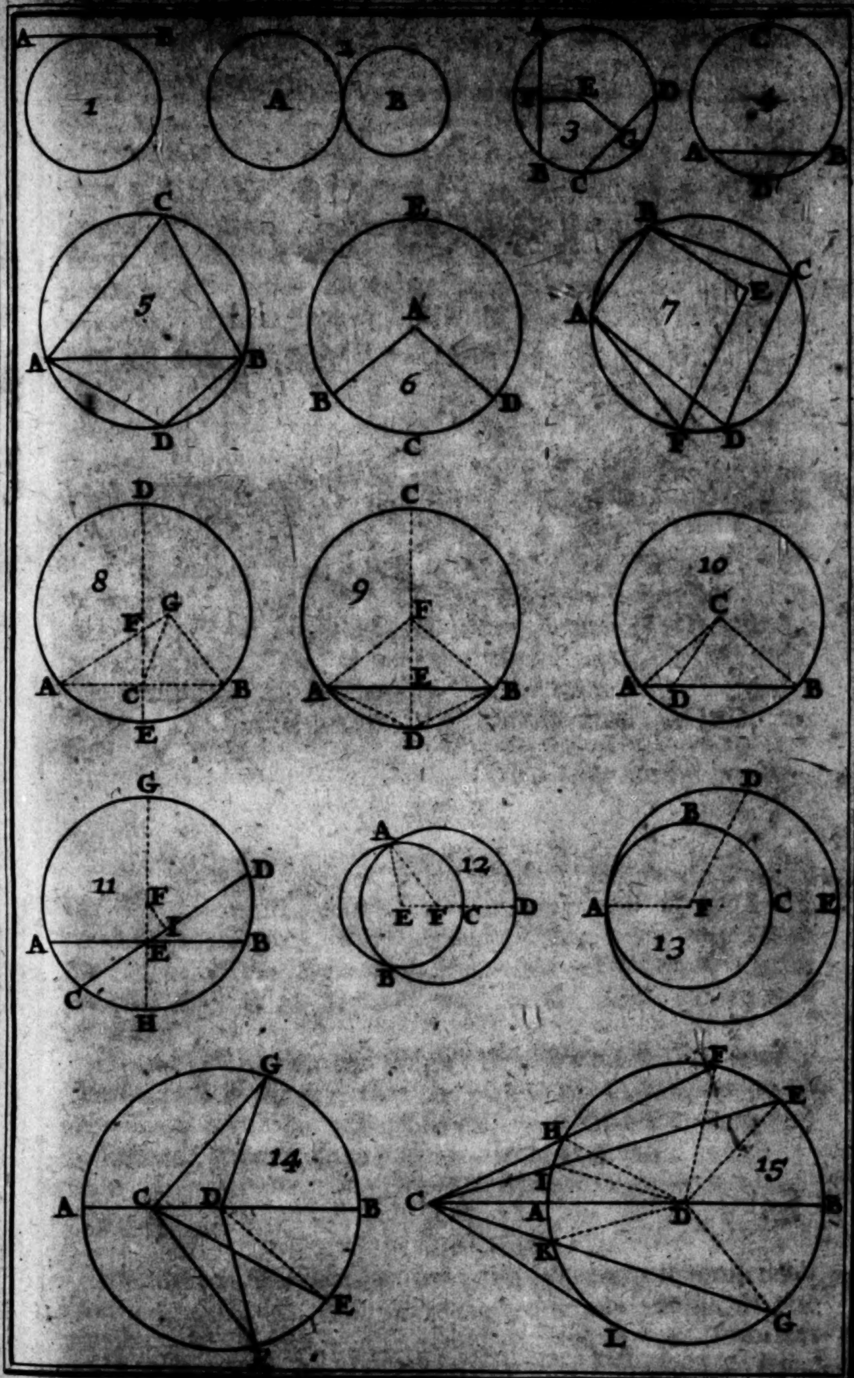
Equal Circles are those whose Diameters, or Semidiameters are equal to each other.

II.

Plate 1. Fig. 1. *A Right-Line* is said *to touch a Circle*, when it meets the Circumference of that Circle without making an Angle with it, that is to say, without cutting it, or without entering within, being produced as AB, and is call'd a *Tangent*.

III.

It is said that *two Circles touches one another*, when their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their Circumferences meet without cutting each other, Fig. 2.
as A and B.

IV.

It is said that *two Right-Lines are equally distant from the Centre of a Circle*, when the two Perpendiculars drawn from the Centre upon those two Lines, are equal to each other. *Thus 'tis known that the two Lines AB, CD, are equally distant from the Centre E, because their Perpendiculars EF, EG, are equal to each other.* Fig. 3.

V.

The *Segment of a Circle*, is a Part of a Circle terminated by a Right-Line and by a Part of the Circumference of the same Circle: *as ABC, or ABD.* Fig. 4.

It is evident that when a Right-Line AB shall pass through the Centre of a Circle, the two Segments ACB, ADB, will be equal to each other, because each will be a Semi-circle. But as we have already said in *Def.* 8. 1. we commonly understand by the Segment of a Circle, a Part of the Circle greater than a Semi-circle, as ACB, or less, as ADB.

VI.

The *Angle of a Segment*, is the mixtilinear Angle form'd by the Circumference of a Circle and the Right-Line, which terminates the Segment. *Thus 'tis said that the Angle of the Segment ACB, is the mix'd Angle BAC; and the Angle of the Segment ADB, is the mix'd Angle BAD, or ABD.* Fig. 5.

It is evident that the Angle of a Segment less than a Semi-circle is Acute, that the Angle of a Segment equal to a Semi-circle is a Right-one, and that the Angle of a Segment greater than a Semi-circle is obtuse.

VII.

The *Angle in a Segment*, is an Angle comprehended by two Right-Lines, which begin from any Point in the Arch of the Segment, and end in the two Extremities of the Right-Line, which serves for the Base to that Segment. *Thus it is said that the Rectilinear Angle ACB is in the Segment ABCA, and that the Rectilinear Angle ADB, is in the Segment ABDA.* Fig. 5.

Fig. 5. It is evident that the Angle ACB, which is in the greater Segment ABCA is less than the Angle ADB, which is in the less Segment ABDA. It is said that the Angle ACB is subtended by the Arch ADB, and that in like manner the Angle ADB is subtended by the Arch ACB. It is also said that a Segment is *capable* of such an Angle, when the Angle in the Segment is equal to that Angle.

VIII.

Similar Segments of a Circle are those which are capable of equal Angles.

It may be said in the same manner that the *similar Arches of a Circle*, are those upon which are form'd equal Angles at the Centre, or at the Circumference: and we call that an Angle *at the Centre* which is made at the Centre of a Circle, or of a Regular Polygon, which is the same as that of the circumscrib'd Circle.

IX.

Fig. 6. The *Sector of a Circle* is the Part of a Circle, terminated by two Semi-diameters, and by a Part of the Circumference of a Circle: *as the Figure ABCD, or the Figure ABED.*

The two Radij AB, AD, must not make one and the same Right-Line, because instead of a Sector would be a Semi-circle. So that a Sector of a Circle is necessarily greater or less than a Semi-circle, as ABCD, or greater as ABED.

X.

Fig. 7. It is said that *a Quadrilateral Figure is inscrib'd in a Circle*, when each of its angular Points touch the Circumference of the Circle, as ABCD.

PROPOSITION I.

PROBLEM I.

To find the Centre of a given Circle.

Fig. 8. TO find the Centre of a Circle, the Circumference whereof is ADBE, draw within any Line whatever as AB, and having divided it equally in two at the Point C, draw through this Point C, the right Line DE, per-

perpendicular to the Line AB; and because in this perpendicular CE, the Centre of the Circle is to be found, there needs no more than to divide it equally in two at the Point F, which will be the Centre required, as we shall demonstrate, by shewing that the Centre of the Circle must be in the Perpendicular DE. Plate 1. Fig. 8.

PREPARATION.

Let us suppose that the Centre of the Circle is G, without considering where that Point G falleth, and let us draw from this Point G, to the two Extremities A, B, of the Line AB, and through its middle Point C, the Right-Lines GA, GB, GC.

DEMONSTRATION.

Because the two Triangles AGC, BGC are equal to each other, by 8. 1. since they have the common side GC, the side GA, equal to the side GB, by *Def. of the Circle*, and the side AC, equal to the side BC, *by constr.* the Angle GCB, will be equal to the Angle GCA, and thus each of its two Angles will be a right one, and consequently equal to the Angle DCB, which is also a right one *by constr.* So that the two Angles DCB, GCB, being equal to each other, the Line CG falleth upon the Line CD, and consequently the Centre G is in CD, or DE. *Which was to be demonstrated.*

COROLLARY.

It follows from this Proposition, that the Centre of a Circle is found in a Right-Line, which divides another Right-Line drawn in the Circle at Right-Angles, and into two equal Parts.

USE.

This Proposition serves for the following ones, which do suppose every where that the Centre of a Circle sought for is found.

Plate I. Fig. 10.

PROPOSITION II.

THEOREM I.

A Right-Line drawn through two Points, taken at pleasure in the Circumference of a Circle, is entirely within the Circle.

I Say that the Right-Line AB, drawn through the two Points A, B, taken at Pleasure in the Circumference of a Circle, the Centre whereof is C, is quite within the Circle: that is to say, that any Point whatever of this Line, as D, is nearer the Centre C, than one of the two Points A, B, which are in the Circumference.

DEMONSTRATION.

Having drawn the Right-Lines CA, CB, CD, it will appear that since the Point C is the Centre of the Circle, the two Lines CA, CB, are equal to each other, and that by 5. 1. the two Angles A, B, are equal to each other; and because the Angle ADC, is exterior with regard to to the Triangle BDC, it is by 16. 1. greater than the interior opposite one B, or than A its equal; wherefore by 19. 1. the side CA will be greater than the side CD, and the Point D, consequently nearer the Center C than the Point A. *Which was to be demonstrated.*

COROLLARY.

It follows from this Proposition, that a Right-Line doth not touch the Circumference of a Circle but in one Point, because if it shou'd touch it in two, it might be drawn from one of those Points to the other, and so wou'd enter within the Circle, and consequently cut its Circumference, and not touch it.

USE.

This Proposition serves for several of the following ones, which suppose that a Right-Line drawn from one Point to another Point of the Circumference of a Circle, falls quite within the Circle; and it is upon this Foundation, one may demonstrate that a Sphere touches a Plane in one Point only.

PRO-

Plate 1.
Fig. 9.

PROPOSITION. III.

THEOREM II.

If the Diameter of a Circle divides into two equal Parts, a Right-Line which passes not through the Centre, it will cut it at Right-Angles; and if it cuts it at Right-Angles, it will divide it into two equal Parts.

I Say first, that if the Diameter CD of the Circle ACBD, cuts the Line AB, which does not pass thro' the Centre F, into two equal Parts in the Point E, each of the two Angles CEA, CEB, will be right ones.

DEMONSTRATION.

If you draw the Radij AF, BF, it will appear by 8. 1. that the two Triangles FEA, FEB, are equal to each other, by reason of the common Side EF, of the Radius AF, equal to the Radius BF, *by Def. of the Circle*, and of the Line AE, equal to the Line BE, *by Sup.* Wherefore the two Angles AEF, BEF, will be also equal to each other; and consequently right ones. *Which was to be demonstrated.*

I say in the second Place, that if the Diameter CD, be perpendicular to the Line AB, so that each of the two Angles which are made at the Point E, be right ones, the Line AB will be divided into two equal Parts in the Point E, that is to say, the Sides AE, BE, of the two Rectangular Triangles AEF, BEF, will be equal to each other, as appears by 26. 1. by reason of the two equal Angles A, B, by 5. 1. and of the common Side EF, similarly posited, or of the Side AF equal to the Side BF.

USE.

This Proposition serves for the Demonstration of *Prop.* 4. 14. & 35. and is us'd in *Trigonometry*, to demonstrate that the Chord of an Arch is double the Sine of the half of that Arch: as here, that the Chord AB, is double the Sine AE, of the Arch AD, which is equal to the half of the Arch ADB, as it may be seen easily by *Prop.* 28. by drawing the Chords AD, BD, which are equal to each

Plate 1. Fig. 9. other, because the Square AD, is by 47. 1. equal to the two Squares AE, DE, or BE, DE, and that the Square BD, is also equal to the same Squares BE, DE, by 47. 1. *&c.* or without referring to *Prop.* 28. it is known that in the equal Triangles, AEF, BEF, the Angles AFE, BFE, are equal to each other, and that consequently the Arches AD, BD, which measure 'em, will be also equal to each other.

PROPOSITION IV.

THEOREM III.

Two Right-Lines cutting each other in a Circle, in one Point which is not its Centre, do not cut one another equally.

Fig. 11. I Say, that if in the Circle ADBC, the Centre whereof is F, the two Right Lines, AB, CD, do intersect in a Point E, different from the Centre F, these two Lines AB, CD, do not cut each other into two equal Parts, that is to say, although the two Parts of the one, as AE, BE, may be equal to each other, the two Parts of the other CE, DE, cannot at the same time be also equal to each other.

DEMONSTRATION.

Since it is suppos'd that the Line AB, is divided equally in two at the Point E, if you draw through this Point E, and through the Centre F, the Diameter GH, the Angle FEB, will be a right one, by *Prop.* 3. wherefore the Angle FED, will be Acute; so that if from the Centre F, the Line FI is drawn perpendicular to the Line CD, this Perpendicular FI, will divide, by *Prop.* 4. the Line CD equally in two at the Point I, which will be different from the Point E. Since then the two Parts CI, DI, are equal to each other, the two CE, DE, will be unequal. *Which was to be demonstrated.*

PRO-

Plate 1. Fig. 12.

PROPOSITION V.

THEOREM IV.

Two Circles which cut each other, have different Centres.

I Say, that the Centers E, F, of the two Circles ABC, ABD, which cut each other in A, are different, so that they do not coincide together.

PREPARATION.

Join the two Centres E, F, by the Right-Line FD, without considering whether this Line FD, be extended and continue it until it cuts the Circumferences of two Circles at the Point CD. Again, imagine the Right-Lines EA, FA, drawn.

DEMONSTRATION.

Because *by Defin. of the Circle*, the Line FA is equal to the Line FD, or FC+CD, and the Line EA to the Line EC, or FC+EF, the Difference of the two Lines FA, EA, will be equal to the Difference of the two FC+CD, FC+EF, that is to say, of the two CD, EF, and because the Line CD is a real one, the Difference of the two Lines FA, EA, will be also real, and the two Centres E, F, will be consequently different. *Which was to be demonstrated.*

SCHOLIUM.

We have chang'd *Euclid's* Demonstration, to a direct one, because the indirect ones do not enlighten the Mind so well. Nevertheless as this Demonstration depends upon some Axioms as yet unmention'd, we shall here explain in few Words *Euclid's* Demonstration, which seems to me more easy for Beginners.

If the two Centres E, F, did coincide together, so that the Centre E, be common to the two Circles ABC, ABD, each of the two Lines EC, ED, wou'd be equal to the same Line EA, *by Def. of the Circle*, and consequently these two Lines EC, ED, wou'd be equal to each other, that is to say, the Part wou'd be equal to the whole, which is absurd, *&c.*

USE.

Plate 1. Fig. 12. This Propoſition ſerves to demonſtrate, that two Circumferences of a Circle cannot cut one another but in two Points, as you will ſee in *Prop.* 10.

PROPOSITION VI.

THEOREM V.

Two Circles which touch one another within, have not one and the ſame Centre.

Fig. 13. I Say, that if the two Circles ABC, ADE, touch at the Point A, they have not one and the ſame Centre, as for Example F.

PREPARATION.

Draw from the ſuppos'd common Centre F, to the Point of Contact A, the Right-Line FA, and another Right-Line whatſoever FD, cutting the Circumference of the great Circle at the Point D, and the Circumference of the little one at the Point B.

DEMONSTRATION.

If the Point F, were the common Centre to the two Circles ABC, ADE, the two Lines FB, FD, wou'd be equal each to the ſame Line FA, and conſequently equal to each other, which is impoſſible, becauſe the Line FD is eſſentially greater than the Line FB. It is therefore impoſſible that the Point F, ſhou'd be the common Centre to the two Circles ABC, ADE. *Which was to be demonſtrated.*

SCHOLIUM.

Euclid demonſtrates this Propoſition only in the Caſe when the two Circles touch one another within, becauſe it is evident, that when they touch without, they cannot have one and the ſame Centre.

USE.

This Propoſition ſerves to demonſtrate *Prop.* 11. & 12. which ſuppoſe that Circles which touch one another within or without, have different Centres.

Plate 1. Fig. 14.

PROPOSITION VII.

THEOREM VI.

If from a Point other than the Centre, taken at pleasure upon the Diameter of a Circle, be drawn several Right-Lines to the Circumference, the greatest of all the Lines is that Part of the Diameter wherein the Centre is, and the least is the remainder of the Diameter. As for the other Lines, the nearest to that which passes through the Centre is greater than another which is more remote from it: and more than two equal Right-Lines cannot be drawn from that same Point, on one Side and the other of the least or of the greatest.

I Say first, that if upon the Diameter AB, you take any where, but on the Centre D, of the Circle AGBF, a Point at pleasure, as C, and if you draw several Right-Lines to the Circumference, as CE, CF, *&c.* the Line CB, wherein the Centre D is found, is the greatest of all, for example greater than the Line CE.

DEMONSTRATION.

Because of the Triangle CDE, the two Sides CD, DE, taken together, are greater than the third CE, by 20. 1. and the two CD, DE, are together equal to the Line CB, by reason of the Radius DE equal to the Radius DB, *by Def. of the Centre,* it follows that the Line CB is greater than the Line CE. *Which was to be demonstrated.* It may be demonstrated in like manner, that the Line CB is greater than the Line CF, and than any other Line, which can be drawn from the Point C.

I say in the second Place, that the Line CA, which is the remainder of the Diameter AB, is the least of all, for example less than the Line CF.

DE-

Plate 1. Fig. 14.

DEMONSTRATION.

By drawing the Radius DF, it will appear as before, that in the Triangle CDF, the two Sides CD, CF, taken together are greater than the third DF, or DA, wherefore if you substract CD from each Side, it will appear that the Line CF is greater than the Line CA. *Which was to be demonstrated.* This also is seen from the following Demonstration.

I say in the third Place, that the Line CE, which is nearer the greatest CB, is greater than the Line CF, which is further from it.

DEMONSTRATION.

Because the two Sides CD, DE, of the Triangle CDE, are equal to the two Sides CD, DF, of the Triangle CDF, and the compris'd Angle CDE is greater than the compris'd Angle CDF, the Base CE will be by 24. 1. greater than the Base CF. *Which was to be demonstrated.*

Lastly, I say that from the same Point C, there cannot be drawn more than two equal Lines to the Circumference, as for example CF, CG, upon supposition that the Angles CDF, CDG, on both Sides are made equal.

DEMONSTRATION.

Because the two Sides CD, DF, of the Triangle CDF, are equal to the two Sides CD, DG, of the Triangle CDG, and the compris'd Angle CDF equal to the compris'd Angle CDG, the Bases CF, CG, will be equal to each other by 4. 1. and as all the Lines which may be drawn on both Sides, will be either nearer CB, or more remote, and consequently greater or less than CF, or CG, it follows that there can be but two equal Lines drawn from it. *Which remain'd to be demonstrated.*

USE.

This Proposition is us'd in *Astronomy*, to demonstrate the different Distances of a Planet from the Earth, and to shew that it is the most distant from the Earth, that it can be, in its *true Apogæum*, and as near the Earth as it can possibly be, in its *true Perigæum*.

PRO-

Plate 1. Fig. 15.

PROPOSITION VIII.

THEOREM VII.

If from a Point taken at pleasure, without a Circle, you draw any Number of Right-Lines, terminating in the Concave Circumference of the Circle, the greatest of all is that which passes thro' the Centre: and that which is nearer it, is greater than another which is further off. On the contrary, of those Lines which fall on the Convex Circumference, that which being produc'd passes through the Center, is the least of all; and that which is nearest it, is less than another which is more remote. Lastly, take it either way, the less or the greater, there can't be drawn from that same Point above two Right-Lines equal to one another.

WE understand by the *Concave Circumference* that which regards the inside, and by the *Convex Circumference*, that which regards the outside. This being suppos'd, I say first, that if from the Point C, taken at pleasure without the Circle AFBG, you draw several Right-Lines meeting the Circumference as well Concave as Convex; the Line CB which passes thro' the Centre D, is the greatest of all those which come to the Concave Circumference, for example greater than than the Line CE.

DEMONSTRATION.

Because by drawing the Radius DE, you have the Triangle CDE, the two Sides whereof CD, DE, are together greater than the third CE, by 20. 1. and because the two Sides CD, DE, are together equal to the Line CB, by reason of the Radius DE equal to the Radius DB, *by Def. of a Centre*; it follows that the Line CB is greater than the Line CE. *Which was to be demonstrated.* In the same manner may be demonstrated that the Line CB is greater than the Line CF, and than any other that shall be drawn from Point C.

I say, secondly, that the Line CE, which is nearer the greatest Line CB, is greater than the Line CF, which is further off.

DE-

Plate 1. Fig. 15.

DEMONSTRATION.

By drawing the Radius DF, it will appear that since the two Sides CD, DE, of the Triangle CDE, are equal to the two Sides CD, DF, of the Triangle CDF, and that the compris'd Angle CDE, is greater than the compris'd Angle CDF, the Base CE will be by 24. 1. greater than the Base CF. *Which was to be demonstrated.*

I say, in the third Place, that the Line CA, which being produc'd passes thro' the Centre D, is the least of those that can be drawn from the Point C to the Convex Circumference, for example less than the Line CI.

DEMONSTRATION.

Because by drawing the Radius DI, you have the Triangle CID, the two Sides whereof CI, DI, taken together, are greater than the Side CD, by 20. 1. by taking away the equal Lines DI, DA, it will be found that the Line CA is less than the Line CI. *Which was to be demonstrated.*

I say, in the fourth Place, that the Line CI, which is nearer to the least Line CA, is less than the Line CH, which is further off.

DEMONSTRATION.

By drawing the Radius DH, it will appear by 21. 1. that the two Sides CI, DI, of the Triangle CID, are together less than the two CH, DH, taken together; wherefore by taking away the equal Sides DI, DH, it is plain that the Line CI is less than the Line CH. *Which was to be demonstrated.*

I say, fifthly, that from the same Point C, you can draw but two equal Lines to the Concave Circumference, for example CE, CG, by supposing there be made on each Side the two equal Angles CDE, CDG.

DEMONSTRATION.

Because the two Sides CD, DE, of the Triangle CDE, are equal to the two Sides CD, DG, of the Triangle CDG, and the compris'd Angle CDE, equal to the compris'd Angle CDG, the Bases CE, CG, will be equal to each

each other by 4. 1. And as all the Lines which can be drawn one Side or the other, will be either nearer to or further from CB, and consequently greater or less than CE, or than CG; it follows that no more than two equal Lines can be drawn from thence. *Which was to be demonstrated.* Plate 1. Fig. 15.

Lastly, I say, that from the same Point C, only two equal Lines can be drawn as far as the Convex Circumference, for example, CI, CK, supposing on each Side the two equal Angles CDI, CDK be made.

DEMONSTRATION.

Because the two Sides CD, DI, are equal to the two Sides CD, DK, and the compris'd Angle CDI of the Triangle CID, equal to the compris'd Angle CDK of the Triangle CKD, the Bases CI, CK, will be equal to each other, by 4. 1. and a third equal one can't be drawn, because according as it approaches more or less to the Line CA, it will be greater or less. *Which remain'd to be demonstrated.*

COROLLARY.

It follows from this Proposition, that the greatest of the Right-Lines that can be drawn from the Point C, to the Convex Circumference of the Circle AFBG, is that which touches this Circumference, as CL, which touches it in L.

PROPOSITION IX.

THEOREM VIII.

The Point from whence three equal Lines may be drawn to the Circumference of a Circle, is the Center of that Circle.

THis is a Consequence from *Prop.* 7. where it has been demonstrated, that from a Point which is not the Center of a Circle, you can't draw to its Circumference more than two equal Lines, and this Proposition is put here only to demonstrate the following.

PRO-

Plate 2. Fig. 16.

PROPOSITION X.

THEOREM IX.

The Circumferences of two Circles intersect only in two Points.

IT is evident that the two Circles ABC, ADC, may cut each other in two Points, as A, C; because if the Point E, is for example, the Centre of the Circle ABC, the Lines EA, EC, drawn from this Centre E, to the Points A, C, will be equal to each other: and as the Point E can't be the Centre of the Circle ADB, by *Prop.* 5. You have another Point E than the Centre of the Circle ADB, from which may be drawn to its Circumference, the two equal Lines EA, EC, which is possible *by Prop.* 7. where we have demonstrated that there can't be drawn from the Point E, to the Circumference of the Circle ADB, more than two equal Lines; from whence it may be concluded, that the two Circles ABC, ADC, can't likewise cut each other in above two Points. *Which was to be demonstrated.*

USE.

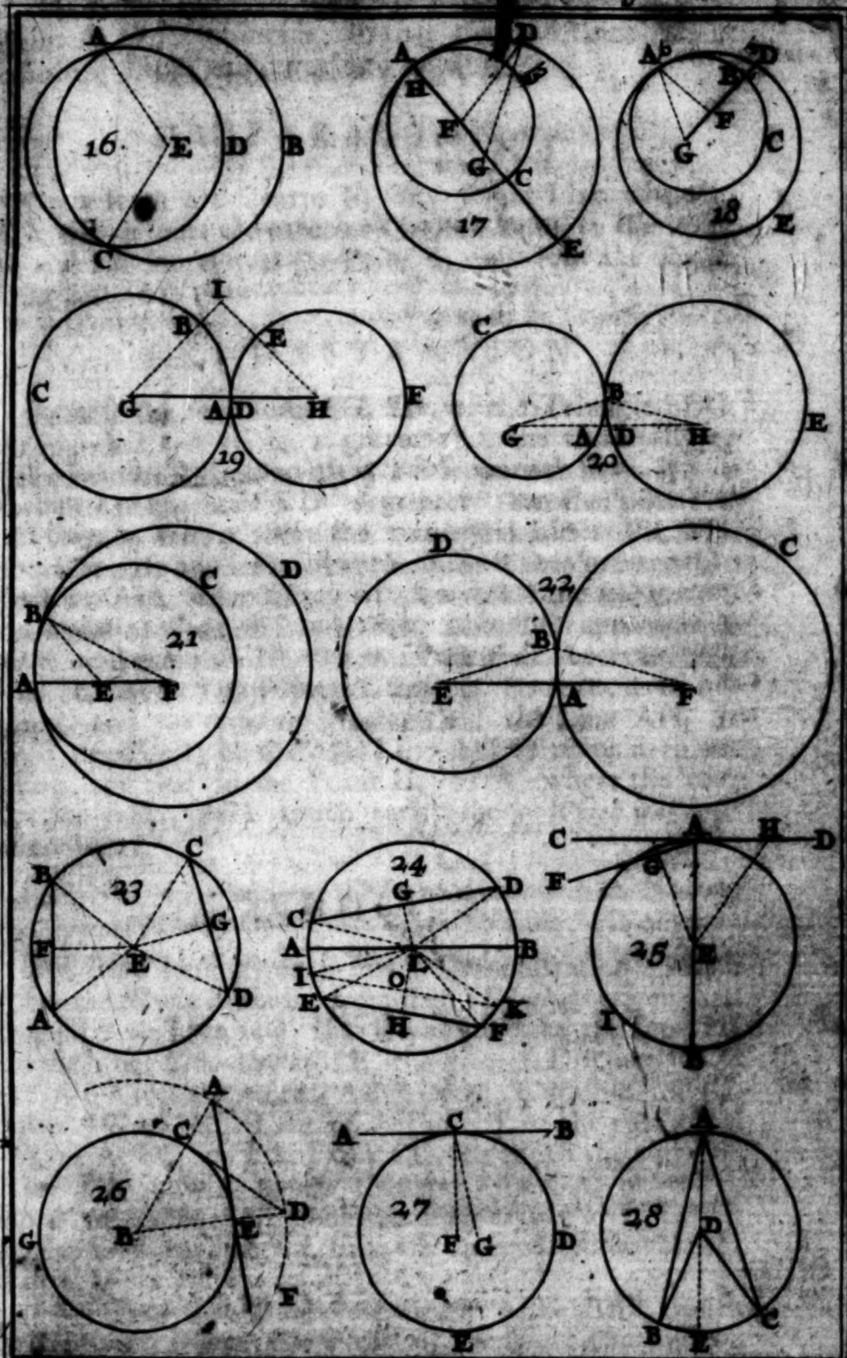
This Proposition serves, as we have already said in *Dechales's Euclid*, to shew that Equations of two Dimensions, which may be all resolv'd by the Intersection of two Circles, have but two Roots, since the Circumferences of two Circles cannot intersect but in two Points.

PROPOSITION XI.

THEOREM X.

If two Circles touch each other within, the Right-Line drawn thro' their Centres, being produc'd, will pass thro' the Point where they touch.

Fig. 17. I Say, that if thro' the Centres F, G, of the two Circles ABC, ADE, whose Circumferences touch each other within, you draw the Right-Line FG, and produce it, till it cuts the exterior Circumference ADE in A, and the interior ABC in H; these two Circles will touch each other in the Points A, H, that is to say, these two Points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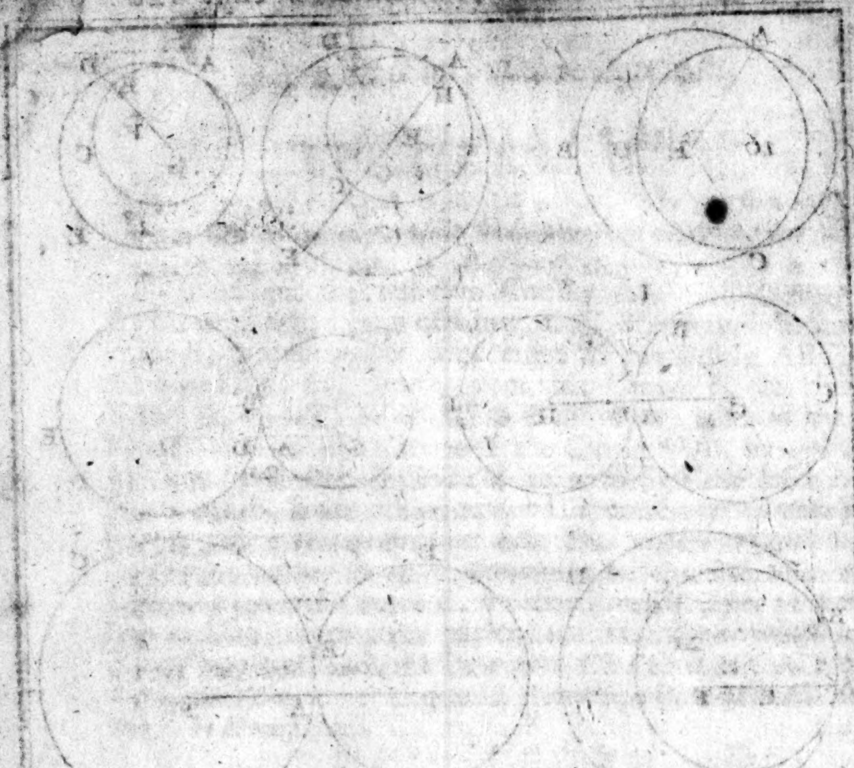

Point A, H, do coincide, so that their Distance AH is infinitely little, and reduc'd to nothing. Plate 2. Fig. 17.

PREPARATION.

Draw from the Centre F, any Right-Line whatever FD, which cuts the exterior Circumference in the Point D, and the interior in the Point B, and join the Right Line BD.

DEMONSTRATION.

Because the two Sides FG, FD, of the Triangle FDG, are together *by Prop.* 20. 1. greater than the third GD, or GA its equal, by taking away FG from each Side, it will appear that the Line FD is greater than the Line FA, and then by taking away the two equal Lines FB, FH, it will at last be found that the Line BD is greater than the Line AH, what distance soever this Line BD is from the Point of Contact: and as the Line BD approaching more and more to the Point of Contact, becomes still less, so that at the Point of Contact 'tis reduc'd to nothing, and yet remains greater than the Line AH, it must necessarily be that this Line AH is reduc'd to nothing, and that in the Point H, or A, where the two Circles ABC, ADE touch each other. *Which was to be demonstrated.*

SCHOLIUM.

We have here given a direct Demonstration, which consequently is different from that of *Euclid*, as you shall see, after we have said, that if you produce the Line FG on the other Side towards E, the greatest Distance CE of the two Circumferences ABC, ADE, is double the Distance FG of their Centres, because if to the two equal Lines FA, FC, or FA, FG+CG, be added the common Line FG, it will appear that the Line GA, or GE is equal to 2FG+CG, wherefore by taking away CG, it will also appear that the Line CE is equal to double the Line FG.

I say then, that if the two Circles ABC, ABE, touch each other within at the Point A, the Right-Line drawn through the Centre F of the Circle ABC, and through the Centre G of the Circle ADE, being continu'd, will pass through the Point of Contact A, so that it cannot go for example to Point D. Fig. 18.

DE-

Plate 2. Fig. 18.

DEMONSTRATION.

For by drawing the Radij FA, GA, it will appear *by* 20. 1. that in the Triangle GFA, the two Sides GF, FA, taken together, that is to say, GF, FB, or the single Line GB is greater than the third Side GA, or GD, which is impossible; it is likewise impossible that the Line FG, being produc'd, should pass through any other Point than the Point of Contact A. *Which was to be demonstrated.*

USE.

Fig. 17. This Proposition serves to describe the Circumference of a Circle, which touches the Circumference of another Circle in a given Point; as if the Point A be given in the Circumference of the given Circle ADE, and you draw from the Centre G, of the given Circle, through the given Point A, the Right-Line AG, upon which you may chuse at pleasure a Point as F, for the Centre of the Circle which will touch in A the propos'd Circle ADE.

PROPOSITION XII.

THEOREM XI.

If the Circumferences of two Circles touch each other without, the Right-Line drawn through their Centres, will pass through the Point where they touch each other.

Fig. 19. I Say, that if thro' the Centres G, H, of two Circles ABC, DEF, whose Circumferences touch each other without, you draw the Right-Line GH, which cuts the Circumference ABC at the Point A, and the Circumference DEF at Point D; these two Circles will touch each other in the Points A, D, that is to say, these two Points A, D, coincide, so that their distance AD is reduced to nothing.

PREPARATION.

Draw thro' the Point I, taken at pleasure without the two Circles ABC, DEF, and thro' their Centres G, H, the Right-Lines GI, HI, which will cut the two Circumferences ABC, DEF, in two Points, as B, E.

DE-

Plate 2. Fig 19.

DEMONSTRATION.

Because the two Sides GI, HI, of the Triangle GHI, are together greater than the third Side GH, by 20. 1. If you take from one Side the two Lines GB, HE, and from the other Side the two GA, HD, which are equal to the two preceding, it will appear that the Sum of the two Lines IB, IE, is greater than the Line AD; and as this Sum becomes less in Proportion as the Point I is nearer to the Point of Contact, so that it is reduc'd to nothing at the Point of Contact, and yet remains greater than the Line AD; this Line AD must necessarily be reduc'd to nothing, and the Point A, or D, be where the two Circles ABC, DEF, touch each other. *Which was to be demonstrated.*

SCHOLIUM.

If this demonstration, which we have render'd direct as much as possibly we cou'd, does not please you, follow that of *Euclid*, which is indirect, as you'll see. Fig. 20.

I say then, if the two Circles ABC, BDE, touch each other without at Point B, the Right-Line GH, drawn thro' the Centres G, H, of those two Circles, will pass thro' the Point of Contact B, so that it can't cut the Circumference ABC, BDE, for example at the two Points A, D.

DEMONSTRATION.

For by drawing the Radii BG, BH, it will be found by 20. 1. that in the Triangle GBH, the two Sides GB, HB, or the two GA, HD, are together greater than the third Side GH, which being impossible, it is likewise impossible for the Right-Line GH, which joins the Centres G, H, of the two propos'd Circles, to pass any where but thro' the Point of Contact. *Which was to be demonstrated.*

USE.

This Proposition and the foregoing serve to demonstrate the following, which supposes that a Right-Line drawn thro' the Centres of two Circles that touch each other, does pass thro' the *Point of Contact*, that is to say, thro' the Point where they touch each other.

Plate 2. Fig. 21.

PROPOSITION XIII.

THEOREM XII.

Two Circumferences of Circles touch each other only in one Point, whether it be within or without.

I Say, first, that if the two Circles ABC, ABD, touch each other within at the Point A, they cannot touch again in another Point, as B.

PREPARATION.

Draw thro' the Centre E of the Circle ABC, to the Centre F of the Circle ABD, the Right-Line EF, which being produc'd will pass thro' the Point of Contact A, *by Prop.* 11. and draw thro' the same Centres E, F, to the other suppos'd Point of Contact B, the right Lines BE, BF.

DEMONSTRATION.

It is known by 20. 1. that in the Triangle BEF, the Sum of the two Sides EB, EF, or EA, EF; or the single Line FA, wou'd be greater than the third Side FB, which being impossible, because FA, FB, are equal Radij, it is also impossible that the two Circles ABC, ABD, which touch each other at the Point A, shou'd touch again at Point B. *Which was to be demonstrated.*

Fig. 22. I say, in the second place, that if the two Circles ABC, ABD, touch each other without, at Point A, they can't touch again in another Point as B.

DEMONSTRATION.

Having made a Preparation like the foregoing, it will be found by 20. 1. that in the Triangle EBF, the Sum of the two Sides EB, FB, or EA, FA, that is to say, the single Line EF, is greater than the third Side EF, which being impossible, it is in like manner impossible that the two Circumferences of the Circles ABC, ABD, which touch each other at the Point A, shou'd again touch at the Point B. *Which was to be demonstrated.*

SCHOLIUM.

There may be added to the Demonstration of each of these two Cases, that if the two Circumferences ABC, ABD, cou'd touch at Point A, and again at Point B, the Right-Line drawn through the Centres F, G, ought by *Prop.* 11, 12. to pass thro' each of these two contact Points A and B, which is impossible. Plate 2. Fig. 21, 22.

PROPOSITION XIV.

THEOREM XIII.

Equal Right-Lines drawn in a Circle, are equally distant from the Centre; and those that are equally distant from the Centre, are equal to each other.

TWo Lines are said *to be in a Circle*, when they are terminated each way in the Circumference, as AB, CD; and I say, first, that if these two Lines AB, CD, are equal to each other, they are equally remote from the Centre E; that is to say, by *Def.* 4. if from the Centre E, be let fall the two Perpendiculars EF, EG, which will divide them equally in two at the Points F, G, by *Prop.* 3. these two Perpendiculars EF, EG, will be equal to each other. Fig. 23.

DEMONSTRATION.

Having drawn the Radij, EA, EB, EC, ED, it will appear by 18. 1. that the two Isosceles Triangles AEB, CED, are equal to each other, and that consequently the two Angles B, C, will be also equal to each other; so that by 26. 1. the two Sides EF, EG, of the two Rectangular Triangles EFB, EGC, are in like manner equal to each other. *Which was to be demonstrated.*

I Say, in the second Place, that if the two Lines AB, CD, are equally remote from the Centre E, that is to say, if their Perpendiculars EF, EG, are equal to each other, these two Lines AB, CD, are likewise equal to each other, which we shall demonstrate, if we shew that their halves BF, CG, are equal to each other.

DEMONSTRATION.

Plate 2. Fig. 23.

Because by 47. 1. the Sum of the Squares BF, EF, is equal to the Square of the Radius BE, or CE, and that in like manner the Sum of the Squares CG, EG, is equal to the Square of the same Radius EC; these two Sums will be equal to each other; wherefore by taking away the equal Squares EF, EG, there will remain the single Square BF equal to the single Square CG, and consequently the Line BF equal to the Line CG, and the double AB equal to the double CD. *Which was to be demonstrated.*

USE.

This Proposition serves to demonstrate, that all the Perpendiculars, let fall from the Centre of a regular Polygon upon each of its Sides, are equal to one another, because this Centre is the same as the Centre of the Circle circumscrib'd, as you will better perceive, when you have read the 4th Book, which treats of regular Polygons, inscrib'd and circumscrib'd round a Circle. We shall likewise make use of this Proposition, to demonstrate a Case of the following; and it may likewise be used to demonstrate that *lesser Circles which are equally distant from the Centre of the Sphere, are equal to each other.*

PROPOSITION XV.

THEOREM XIV.

If several Right-Lines be drawn in a Circle, the greatest of all is the Diameter, and that which is nearest the Centre, is greater than that which is further off.

Fig. 24.

I Say first, that the Diameter AB of the Circle, whose Centre is L, is the greatest of all other Right-Lines that can be drawn in this Circle, for example greater than the Line CD, which is not a Diameter.

DEMONSTRATION.

If you draw the two Radij LC, LD, then by 20. 1. in the Triangle CLD, the Sum of the two Sides LC, LD, or LA, LB, that is to say, the Line AB, is greater than the third Side CD. *Which was to be demonstrated.* In the same

ſame manner 'tis demonſtrable that the Diameter AB is greater than any other Line whatever, that can be drawn in the Circle thro' a Point which is not the Center. Plate 2. Fig. 24.

I ſay, in the ſecond Place, that the Line EF, which is more remote from the Centre L, than the Line CD, is leſs than that Line CD, which is nearer it.

PREPARATION.

Draw from the Centre L, the Line LG, perpendicular to the Line CD, and the Line LH perpendicular to the Line EF; and as this Line LH is greater than the Line LG, becauſe its ſuppos'd that the Line EF, is further from the Centre L, than the Line CD, take the Line LO equal to the Line LG, and draw thro' the Point O, in the Line LH, the Perpendicular IK, which will be equal to the Line CD, by *Prop.* 14. Laſtly, Draw the Radij LI, LK, LE, LF.

DEMONSTRATION.

Becauſe the two Sides LI, LK, of the Triangle ILK, are equal to the two Sides LE, LF, of the Triangle ELF, and that the compris'd Angle ILK, is greater than the compris'd Angle ELF, the Baſe IK, or CD its equal, will be greater than the Baſe EF, by *Prop.* 24. 1. *Which remain'd to be demonſtrated.*

USE.

This Propoſition ſerves to demonſtrate in the Sphere, that the ſmall Circles which are further off, from the Centre of the Sphere, are leſſer, becauſe their Diameters are leſſer.

Plate 2. Fig. 75.

PROPOSITION XVI.

THEOREM XV.

The perpendicular Line drawn thro' the Extremity of the Diameter of a Circle, is wholly without the Circle; and every other Right-Line drawn between it, and the Circumference of the Circle cuts it, and enters within it.

I Say, first, that if thro' the extremity A of the Diameter AB, of a Circle whose Center is E, you draw the Line CD, perpendicular to the same Diameter AB; that Perpendicular CD is quite out of the Circle, so that any Point whatever of this Perpendicular CD, as H, is more remote from the Centre E than the Point A.

DEMONSTRATION.

If you draw the Right-Line EH, you will have the Rectangular Triangle EAH, whose Hypotenuse EH is greater than the Side EA, by 19. 1. because it is opposite to the Right-Angle A, which is the greatest by 32. 1. Whence it follows that the Point H, is further from the Centre E than the Point A, which is in the Circumference, and that consequently the Line CD is quite without the Circle, so that it touches the Circle in the Point A. *Which was to be demonstrated.*

I say, secondly, that from the Point of Contact A, there can't be drawn below the Tangent CD, any Right-Line, for instance AF, which does not cut the Circumference of the Circle; and which does not enter into it.

PREPARATION.

Let fall from the Centre E, on the Line AF, the Perpendicular EG, which will cut the Line AF in some Place below the Point A, as in E, by reason of the acute Angle EAF.

DEMONSTRATION.

Because the Angle G is right, it will be the greatest of the Angles of the Triangle EGA, by 32. 1. and by 19. 1. the Hypothenuse EA will be greater than the Side EG, Whence it follows that the Point G is nearer the Centre

Centre E than the Point A, and so the Line AF, cuts the Circle, and enters it. *Which remain'd to be demonstrated.* Plate 2. Fig. 25.

SCHOLIUM.

The Commentators of *Euclid* add to this Proposition, that the *Angle of the Semi-circle*, namely, that which the Diameter of a Circle makes with its Circumference, as EAI, is greater than any Rectilineal Acute Angle whatever; which is evident from our Definition of the Angle, by the which it is known that the mix'd Angle EAI is equal to the right-lin'd Angle EAC, which is a right one.

They add likewise, tho' unnecessarily, that CAI, which they have very improperly call'd *Angle of Contact*, is less than any right-lin'd Angle whatever, and that consequently it is reduc'd to nothing, which is likewise evident, because that is not an Angle, as we have observ'd in *Def.* 9. 1.

USE.

This Proposition serves for *Prop.* 33. and likewise to draw a Tangent thro' a Point given in the Circumference of a given Circle; as if the Point A be given, you must draw thro' this Point A, to the Centre E, the right AE, to which on the same Point A erect the Perpendicular AD, which will be the Tangent requir'd. We shall teach in the following Proposition the manner of drawing a Tangent, thro' a Point given without the Circle.

PROPOSITION XVII.

PROBLEM II.

From a given Point without a given Circle, to draw a Right-Line which touches its Circumference.

TO draw from the given Point A, without the given Circle ECG, whose Centre is B, a Right-Line, which touches the Circumference ECG; Draw thro' the given Point A, to the Centre B, the Right-Line AB, which here cuts the Circumference ECG, in the Point C, through which draw to the Line AB, the indefinite Perpendicular CD, which will be terminated in D, by the Circumference of a Circle describ'd from the Center B, thro' the given Point A. Lastly, draw from the Center B, thro' the Point D, the right BD, and thro' the Point E, where it cuts the Circumference ECG, draw to the giv'n Point A, the right AE, which will be the Tangent requir'd. Fig. 26.

Pla. 2. Fig. 26.

DEMONSTRATION.

It is plain by 4. 1. that the two Triangles BAE, BDC, are equal to each other, because they have the two Sides BA, BE, equal to the two Sides BD, BC, and the common compris'd Angle B, wherefore the Angle BEA will be equal to the Angle BCD, which being right, the Angle BEA will be also right, and by *Prop.* 16. the Right-Line AE will touch the Circle ECG in the Point E. *Which was to be done and demonstrated.*

USE.

The Use of Tangent Lines is very frequent in *Trigonometry*, as well *Spherical* as *Rectilineal*; as also in *Dioptricks*, to determine the points of Reflexion upon a curved Surface, as well Concave as Convex. 'Tis likewise made use of in *Dyalling*, for the Description of the *Babylonian* and *Italian* Hours; and in *Navigation*, where we take a Tangent-Line for our Horizon when we observe the Height of the Sun, or some other Star. 'Tis also very commodiously made use of in *Speculative Geometry*, for the *Quadrature of Curves*, whereof you have an Example in the first Theorem of our *Planimetry*, which will serve for the *Quadrature of the Circle*, and of the *Parabola* We shall lay down in *Prop.* 31. another more easie Method to draw Tangents.

PROPOSITION XVIII.

THEOREM XVI.

A Right-Line drawn from the Centre of the Circle, to a Point where another Right-Line touches its Circumference, is perpendicular to that other Right-Line.

Fig. 25. I Say, that if the Right-Line CD, touches in the Point A, the Circumference of the Circle AIB, whose Centre is E; the Right-Line AE drawn thro' the Point of Contact A, and thro' the Centre E, is perpendicular to the Tangent-Line CD.

DEMONSTRATION,

For if the Line EA is not perpendicular to the Tangent-Line CD, it will make with it on the one Side an

an Acute-Angle, and on the other an obtuse one: if for example you wou'd have the Angle EAC obtuse, you may cut off the Right-Angle EAF, by the Line AF, which in this case being perpendicular to the Diameter AB, will touch the Circle at the same Point A, where 'tis suppos'd that the Line CD touches it by *Prop*. 16. and so being quite out of the Circle, you may draw between the Tangent-Line AC, and the Circumference AIB, a Right-Line, which is contrary to the second Case of the *Prop*. 16. Therefore there is no other Line perpendicular to the Diameter AB, than the Tangent Line CD. *Which was to be demonstrated.* Plate 2. Fig. 25.

SCHOLIUM.

This Proposition may yet be demonstrated several other ways, among the rest I have chosen the following, which seems to me the plainest and easiest of all.

If the Line EA is not perpendicular to the Tangent-Line CD, let it be EH, so that the Angle H be a right one, in which case this Angle H will be the greatest of the three Angles of the Triangle EAH, by 32. 1. and by 19. 1. the Side EA will be greater than the Side EH, and the Point H will be within the Circle, and so the Line CD will not be a Tangent-Line. There is not therefore any other Line perpendicular to the Tangent-Line CD, but the Diameter AB. *Which was to be demonstrated.*

This Demonstration is not direct, but it may be made direct, by saying that since the Line CD touches the Circumference AIB, at the Point A, all its Points are further distant from the Centre E than the Point A, and thus all the Right-Lines which shall be drawn from the Centre E, thro' all these Points, will be larger than the Line EA, the which being the shortest of all, ought to be perpendicular to the Tangent-Line CD, by 8. 1, *&c.*

USE.

This Proposition serves for the Demonstration of the following, and likewise of *Prop*. 32 and 36.

Plate 2.
Fig. 27.

PROPOSITION XIX.

THEOREM XVII.

A Perpendicular drawn to a Right-Line which touches a Circle, at the Point of Contact, passes thro' the Centre.

I Say, that if the Line AB, touches at the Point C, the Circumference of the Circle CDE, and if thro' the Point of Contact C, be drawn the Right-Line CF perpendicular to the Tangent AB, the Centre of the Circle CDE is in the Perpendicular CF, or which is the same thing, this Perpendicular CF passes thro' the Centre.

DEMONSRATION.

For if it is suppos'd that the Centre of the Circle is in G, and that you draw the Right GC, it will be perpendicular to the Tangent AB, by *Prop.* 18. and because the Right-Line CF is also perpendicular to the Tangent AB, by *Sup.* the two Angles BCF, BCG, being right ones, will be equal to each other, and the Line CG, will consequently agree with the Line CF. Whence it follows that the Centre of the Circle will be in the Line CF, *Which was to be demonstrated.*

PROPOSITION XX.

THEOREM XVIII.

The Angle at the Centre is double the Angle at the Circumference of a Circle, when these two Angles stand on one and the same Arch.

Fig. 28.

THE *Angle at the Circumference*, so call'd, is that whose forming Lines are in a Circle, and whose angular Point is in the Circumference of the same Circle, as BAC, one of whose Sides may be in a Right-Line with the Sides of the Angle at the Centre BDC, as in *Fig.* 29. Or its two Sides may inclose the Angle at the Centre, as in *Fig.* 28. Or one of its two Sides may cut one of the two Sides of the Angle at the Centre, as in *Fig.* 30. In

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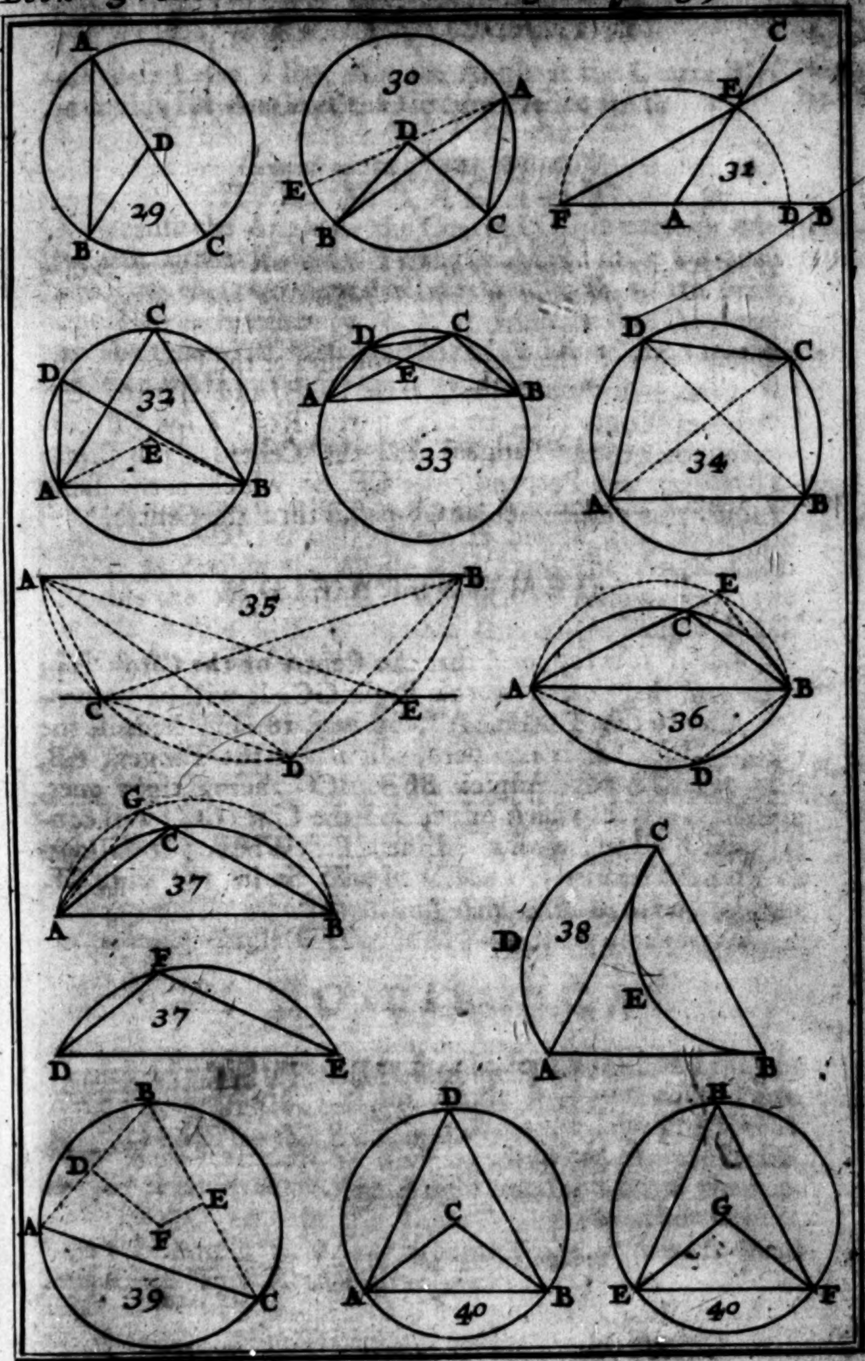
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all these Cases, I say, that the Angle at the Centre BDC is double the Angle at the Circumference BAC. Plate 2. Fig. 28.

Demonstration of the first Case.

Because the Angle in the Centre BDC is exterior with respect to the Isosceles Triangles ADB, it is by 32. 1. equal to the two opposite interiors A, B, which being equal to each other by 5. 1. it follows that the Angle at the Centre BDC is double the Angle at the Circumference BAC. *Which was to be demonstrated.* Plate 3. Fig. 29.

Demonstration of the second Case.

Having drawn from the Angle A, thro' the Centre D, the right ADE, it will appear as before, that the Angle BDE is double the Angle BAE, and the Angle CDE double the Angle CAE. Whence it follows that the whole Angle BDC is double the whole Angle BAC. *Which was to be demonstrated.* Plate 2. Fig. 28.

Demonstration of the third Case.

Having in like manner drawn the right ADE, it will also be found as before, that the Angle BDE, is double the Angle BAE, and that the whole Angle CDE, is double the whole Angle CAE. Whence 'tis easy to conclude that the remaining Angle CDB, is double the remaining Angle CAB. *Which was to be demonstrated.* Plate 3. Fig. 30.

USE.

Fig. 31.

This Proposition serves for the following, and may be of use in dividing a given Angle into two equal Parts, as BAC, to wit, by describing from the angular Point A, the Semi-circle DEF, and by drawing the right EF, which will make at F an Angle equal to half of the propos'd BAC, because the Angle A is made at the Centre, and the Angle F at the Circumference, and both stand upon the same Arch DE.

PRO.

PROPOSITION XXI.

THEOREM XIX.

Plate 3.
Fig. 32, 33. *The Angles which are in one and the same Segment of a Circle, are equal to each other.*

THere may happen two Cases, because the Angles may be in a Segment greater than a Semi-circle, or in a Segment less than a Semi-circle. They may likewise be in a Semi-circle; but this third Case will be demonstrated as the second; wherefore we shall speak only of the two first.

Fig. 32. I say therefore, first, that the two Angles D, C, which are in the Segment ABCD, greater than a Semi-circle, are equal to each other.

DEMONSTRATION.

By drawing from the Centre E, the two Radij, EA, EB, it will appear by *Prop.* 20. that each of the two Angles at Circumference C, D, is equal to half of the Angle at the Centre AEB, and that consequently these two Angles C, D, are equal to each other. *Which was to be demonstrated.*

Fig. 33. I say, in the second Place, that the two Angles C, D, which are in the Segment ABCD, less than a Semi-circle, are equal to each other.

DEMONSTRATION.

Because the two Angles CAD, CBD, are in the Segment CBAD greater than a Semi-circle, they are equal to each other by the preceding Case; and because the two opposite and vertical Angles AED, BEC, are also equal to each other, by 15. 1. it follows by 32. 1. that the Angles ACB, ADB, are equal to each other. *Which remain'd to be demonstrated.*

USE.

USE.

As it is taken for a Principle in *Optics*, That a Line appears always equal, when it is seen under equal Angles, it is manifest from this Proposition, that the Line AB ought to appear equal, being seen from the Points C, D, or any other Point whatever of the Arch ADCB, since thus it is always seen under equal Angles. Plate 3. Fig. 34.

This Proposition serves also for the following; and to describe a great Circle whose Centre cannot be had, which is extreamly useful in the Description of great Astrolabes, which are made by the Principles of the Stereographical Projection of the Sphere; and likewise to give a Spherical Figure to Copper Tools, on which Glasses for Telescopes are to be ground and polish'd. This great Circle is describ'd mechanically thus.

To describe for example, a Circumference of a Circle, thro' the three given Points A, B, C, you are to form upon Iron, or some other solid Matter, an Angle ACB, equal to that which contains the Segment ABCD, and having put in the Points A, B, two Iron-Pins very firm, you must move the Triangle ACB, the Sides whereof CA, CB, ought to be sufficiently long, so that the Side CA touches the Pin A, and the Side CB the Pin B, and then the Point A will describe by this Motion the Circumference ADCB.

Because the Inverse of this Proposition is likewise true, it may be of very good use to draw through a given Point a Line parallel to a given inaccessible Line on the Ground, as you shall see.

Through the given Point C, to draw a Line CE parallel to an inaccessible given Line AB upon the Ground, measure with a Graphometre, or otherwise, the Angle ACB, and choose upon the Ground the Point D, so that the Angle ADB be equal to the Angle ACB, to the end that the four Points A, C, D, B, be in a Circumference of a Circle. After that, make at the Point C, with the Line CB, the Angle BCE equal to the Angle ADC, draw the Right-Line CE, which will be parallel to the given Line AB, by 29. 1. because the Angle BCE is equal to its alternate Angle ABC, equal by *Prop.* 21. to the Angle ADC, since each stands on the same Arch AC, *&c.* Fig. 35.

PRO-

Plate 3. Fig. 34.

PROPOSITION XXII.

THEOREM XX.

The two opposite Angles of a Quadrilateral Figure inscrib'd in a Circle, are taken together equal to two Right-Angles.

I Say, that the two opposite Angles BAD, BCD, of the Quadrilateral ABCD inscrib'd in a Circle, are taken together equal to two right ones, that is to say, they are equal to the three Angles of a Triangle, namely of the Triangle BCD, which taken together are equivalent to two right ones, by 32. 1.

DEMONSTRATION.

If you draw the two Diagonals AC, BD, it will appear by *Prop.* 21. that the Angle BDC, is equal to the Angle BAC, which stands upon the same Arch BC, and that in like manner the Angle DBC is equal to the Angle DAC, which stands upon the same Arch CD: Whence it follows that the whole Angle BAD is equal to the Sum of the two Angles BDC, DBC; wherefore by adding the common Angle BCD, it will appear that the Sum of the two opposite Angles BAD, BCD, is equal to the Sum of the three BDC, DBC, BCD, that is to say, to two right ones. *Which was to be demonstrated.*

SCHOLIUM.

To be the more convinc'd of the Truth of this Theorem, you may consider that since by *Prop.* 20. the Angle at the Circumference is but half the Angle at the Centre, which is measur'd by the Arch that subtends these two Angles, it follows that the Angle at the Circumference BAD, contains but half the Degrees of the Arch BCD, and that in like manner, the Angle BCD contains but half the Degrees of the Arch BAD, and that consequently these two Angles BAD, BCD, contain together but half the whole Circle, or 360 Degrees, that is to say, they make together 180 Degrees, or two Right-Angles. *Which was to be demonstrated.*

USE.

This Proposition serves to demonstrate Part of *Prop.* 31 and 32.

PRO-

Plate 3. Fig. 36.

PROPOSITION XXIII.

THEOREM XXI.

Two similar Segments of a Circle, describ'd on one and the same Right-Line, are equal to each other.

I Say, that if the two Segments of a Circle ABCA, ABDA, are alike, so that they comprehend the equal Angles ACB, ADB, they will be equal to each other.

PREPARATION.

Imagine the Segment ADB, applied on the Segment ACB, turning it towards C, round the common Base AB; and then you will find that these two Segments do not exceed each other; that is to say, the Circumference ADB will fall no where but on the Circumference ACB; and if you wou'd have it reach AEB, produce the Line AC as far as E, and join the Right-Line BE.

DEMONSTRATION.

Since you wou'd have the Segment AEB, be the same as the Segment ADB, which is suppos'd equal to the Segment ACB, the Segment AEB must too be equal to the Segment ACB; and consequently the Angle E be equal to the Angle ACB, *per Def.* 8. which being impossible, because the Angle ACB exterior, is greater than the opposite interior E, by 16. 1. it is also impossible that the Segment ADB should fall any where but on the Segment ACB. Whence it follows that the two Segments ACB, ADB, are equal to each other. *Which was to be demonstrated.*

PRO-

Plate 3. Fig. 37.

PROPOSITION XXIV.

THEOREM XXII.

Two like Segments of a Circle, describ'd upon two equal Lines, are equal to each other.

I Say, that if the two Bases AB, DE, of the two Segments of a Circle, ABCA, DEFD, are equal to each other, and that these two Segments be alike, so that they contain the equal Angles ACB, DFE; these same two Segments ABC, DEF, will be equal to each other.

PREPARATION.

Imagine the Segment DEF lay'd upon the Segment ABC, so that the Base DE coincides with the Base AB; which is possible because these two Bases are suppos'd equal: and then you will find that these two Segments will not exceed each other; that is to say, they will coincide, and if you would have the Segment DEF take up the Space AGB, produce the Line BC as far as G, and join the right AG.

DEMONSTRATION.

Since you wou'd have the Segment AGB to be the same as the Segment AEF, which is suppos'd equal to the Segment ACB, the Segment AGB must likewise be equal to the Segment ACB, and consequently the Angle G be equal to the Angle ACB, by *Def.* 8. which being impossible, because the exterior Angle ACB is greater than the opposite interior one G, by 16. 1. it is also impossible that the Segment DEF shou'd fall any where but on the Segment ACB. From whence it follows that the two Segments ABC, DEF, are equal to each other. *Which was to be demonstrated.*

USE.

Fig. 38. This Proposition is made use of to reduce a mix'd Isosceles Triangle, whose two equal Sides are two Arcs of equal Circles, into a Rectileneal Isosceles Triangle: As if the propos'd Triangle be ADCEB, whose two Sides ADC, BEC, are two equal Arcs of equal Circles, you are to draw the Right-Lines AC, BC, the which

which with the Base AB, will make the Rectilineal Isosceles Triangle ABC, equal to the propos'd ADCEB, because of the two equal Segments of a Circle, ACD, BCE, *&c.*

PROPOSITION XXV.

PROBLEM III.

A Segment of a Circle being given, to find the Centre of that Circle.

TO find the Centre of a Circle, whose Segment is ABC; choose at pleasure three Points upon the Circumference ABC, as A, B, C, and join the Right-Lines AB, BC, and having divided them equally in two at the Points D, E, erect on those Points the two Perpendiculars DF, EF, and their Point of Intersection F, will be the Centre sought. Plate 9. Fig. 32.

DEMONSTRATION.

Because by *Prop.* 1. the Centre of the Circle, whose Circumference passes thro' the three Points A, B, C, is in each of the two Perpendiculars DF, EF, it ought to be in their common Intersection F, where consequently the Centre of the Circle must be, whereof ABC is a Segment. *Which was to be done and demonstrated.*

USE.

This Proposition is the Foundation of the Practice which we have taught in the Resolution of *Probl.* 22. *Introd.* and it likewise serves *to describe the Circumference of a Circle, thro' the three angular Points of a given Triangle*, as will be taught in *Prop.* 4. 5.

PROPOSITION XXVI.

THEOREM XXIII.

In equal Circles, the equal Angles at the Centre, or at the Circumference, are subtended by equal Arches.

Fig. 40. I Suppose that the Circles ABD, EFH, are equal, so that the Radij CA, GE, be equal to each other. This being so, I say, first, that if the Angles at the Centre ACB, EGF, are equal to each other; the Arches AB, EF, which subtend them, are in like manner equal to each other, because they are their Measures.

I say, secondly, that if the Angles at the Circumference D, H, are equal to each other, the Arches AB, EF, on which they stand, are likewise equal to each other, because by *Prop.* 20. those Angles D, H, are the halves of the Angles at the Centre C, G, which are equal to each other, and consequently have their equal Measures AB, EF. *Which was to be demonstrated.*

PROPOSITION XXVII.

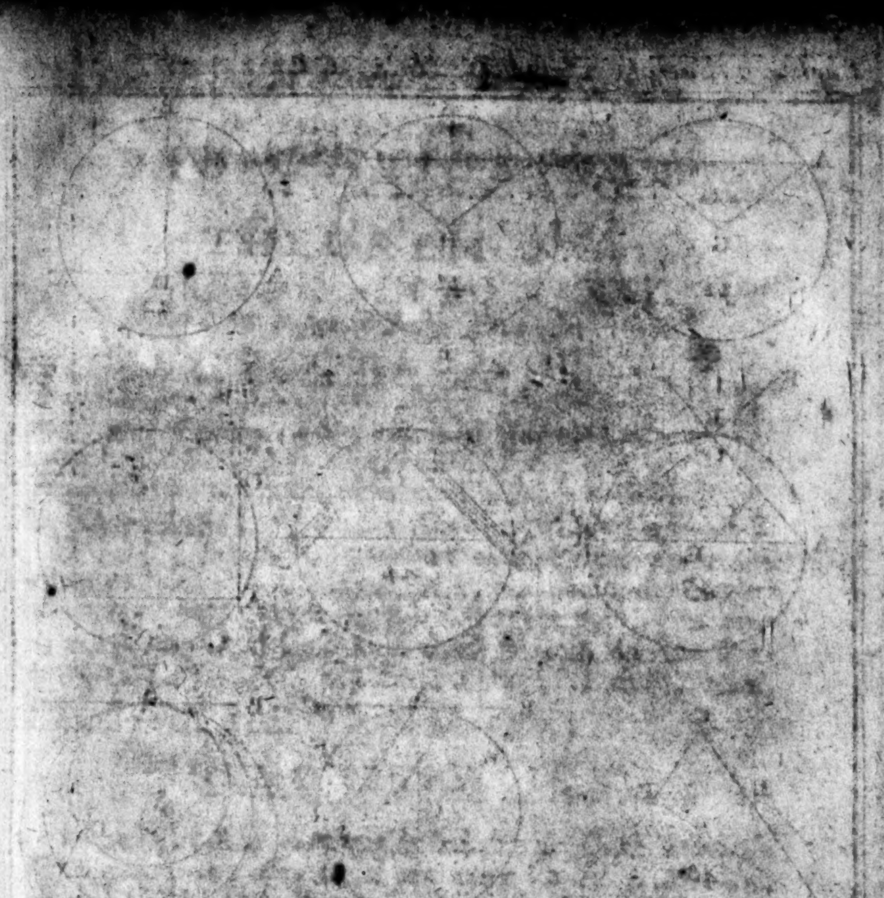
THEOREM XXIV.

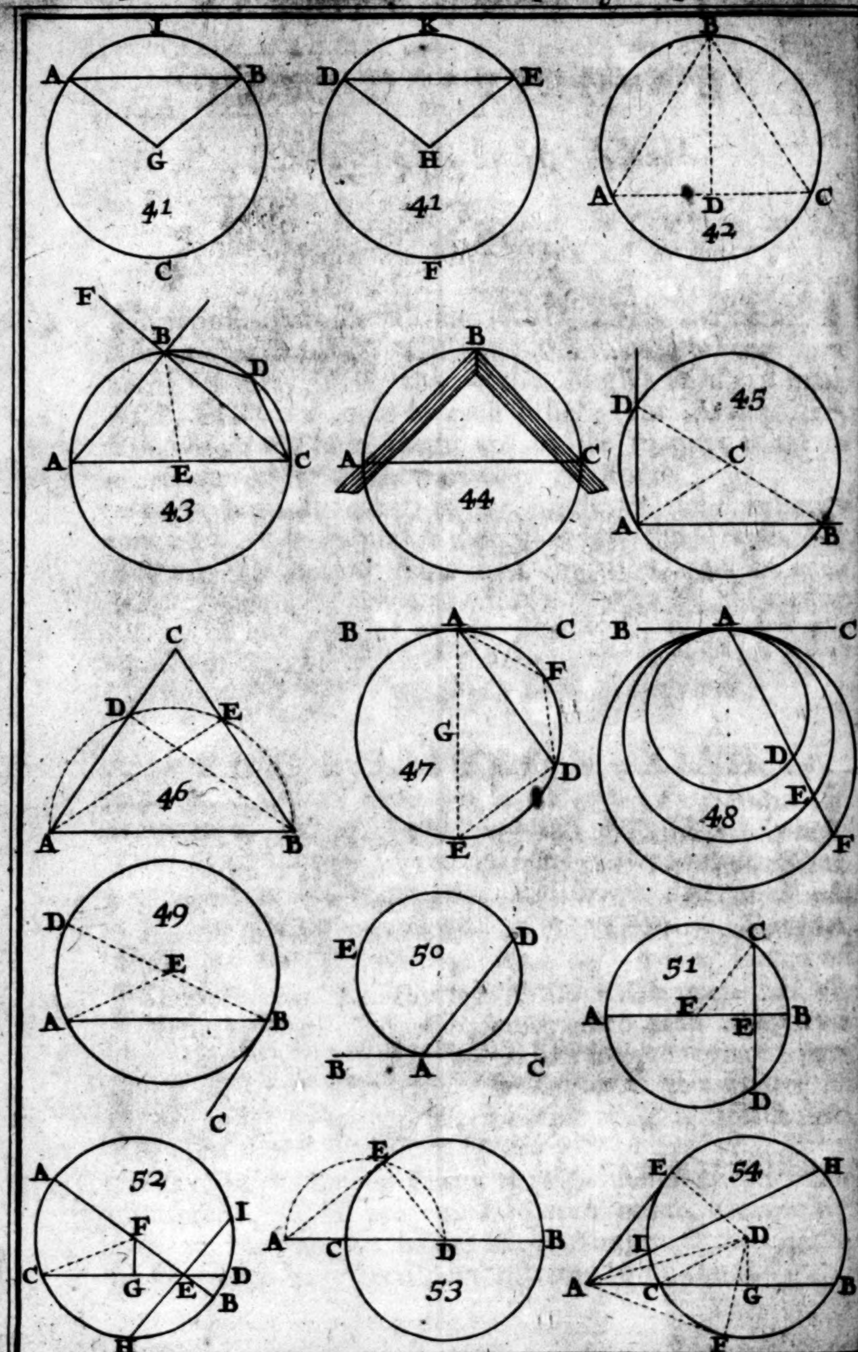
The Angles at the Centre or Circumference of equal Circles, are equal to each other, when they are subtended by equal Arches.

Fig. 40. I Suppose that the Circles ABD, EFH, are equal, so that the Radij CA, GE, be equal to each other, and that the Arches AB, EF, are in like manner equal. This being so, I say, first, that the Angles at the Centre C, G, are equal to each other, because their Measures AB, EF, are suppos'd equal.

I say, in the second Place, that the Angles at the Circumference D, H, are equal to each other, because by *Prop.* 20. they are the halves of the Angles, at the Centre C, G, which have been demonstrated to be equal.

PRO-

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C F A E B D
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A F I C G E D B H
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E A C D B
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E H I D A C G B F

PROPOSITION XXVIII.

Plate 4. Fig. 41.

THEOREM XXV.

Equal Lines in equal Circles have equal Arcs.

I Suppose the Circles ABC, DEF, are equal, and consequently their Radij AG, DH, as also the Lines AB, DE; then, I say, the Arcs AIB, DKE, are equal, because they are the Measures of the two Angles at the Centre G, H, but they by 8. 1. are equal. *Which was to be demonstrated.*

PROPOSITION XXIX.

THEOREM XXVI.

Right-Lines subtending equal Arcs in equal Circles, are also equal,

I Suppose the Circles ABC, DEF, are equal, consequently their Radij AG, DH, and the Arcs AIB, DKE; then, I say, the Lines AB, DE, are equal, for the Arcs AIB, DKE being suppos'd to be equal, the Angles at the Centre G, H, measured by them must also be equal, and by 4. 1. the Isosceles Triangles, ABG, DEH, are equal, and consequently their Bases AB, DE. *Which was to be demonstrated.* Fig. 41.

Plate 4.
Fig. 42.

PROPOSITION XXX.

PROBLEM IV.

To bisect a given Arc.

TO bisect the Arc ABC, join the two Extremities, A, C, by the Right-Line AC, and bisecting it in the Point D, let fall the Perpendicular BD, and that will bisect the Arc propos'd ABC, so that the two Arcs AB, BC, shall be equal.

DEMONSTRATION.

Drawing the Lines AB, BC, you will find by 4. 1. they are equal, the right-angled Triangles ADB, CDB being equal. Consequently by *Prop.* 28. the two Arcs AB, BC, are also equal. *Which was to be demonstrated.*

USE.

This Proposition serves to bisect an Angle, divide a Circle into 32 equal Parts, for the 32 Winds or Points of the Nautical Compass. It serves also to divide a Circle into its 360 Degrees, tho' 'tis but in Part, because we should know how to divide a Circle at least into three equal Parts, which can't be done by the common Geometry, it being a solid Problem, but in practice we are contented with making this Division by Tentation, which is enough for coming at what is propos'd to be effected.

PRO-

PROPOSITION XXXI.

THEOREM XXVII.

In a Circle, an Angle in a Semi-circle is right, that in a greater Segment is acute, that in a less, is obtuse.

I Say first, the Angle ABC, in the Semi-circle ABDC is right, so that producing one of the Lines BA, BC, for instance BC towards F, the Angles ABF, ABC, will be equal, consequently right. Plate 4. Fig. 43.

DEMONSTRATION.

Draw the Radius BE, and by 5. 1. you know that in the Isosceles Triangle AEB, the Angle ABE is equal to the Angle BAE, and in like manner in the Isosceles Triangle BEC, the Angle EBC is equal to the Angle BCE. Whence it follows, that the whole Angle ABC is equal to the sum of BAC, BCE, that is to say by 32. 1. to the external Angle ABF, and consequently each of the two Angles ABC, ABF is right. *Which was to be demonstrated.*

I say, in the second Place, that the Angle BAC, in the Segment BAC, greater than a Semi-circle, is acute, or less than a right.

DEMONSTRATION.

Since the Triangle ABC is right angled in B, as has been demonstrated, it follows by 32. 1. that each of the other two Angles are acute, consequently that BAC is less than a right. *Which was to be demonstrated.*

Lastly, I say, the Angle D, in the Segment BCD, less than a Semi-circle, is obtuse or greater than a right.

DEMONSTRATION.

Because the two opposite Angles A, D of the Quadrilateral Figure ABDC, are taken together equal to two right ones, by *Prop.* 22. and the Angle A has been demonstrated to be acute, the Angle D must be obtuse. *Which was to be demonstrated.*

USE.

Plate 4. Fig. 44. This Propoſition ſerves to find whether a Square be true; for deſcribing the Semi-circle ABC, and applying the right Angle of the Square to any Point of the Circumference, for inſtance B, that one of its Legs, as AB, touch the Extremity A of the Diameter AC, if the other Leg BC alſo touch the other Extremity C, the Square is juſt.

Fig. 45. This Propoſition is alſo very uſeful in erecting a Perpendicular upon a given Point of a given Line: Thus if you were to erect a Perpendicular upon the Point A of the given Line AB, deſcribe thro' the given Point A, upon the Point C, taken at Diſcretion without the given Line AB, the Circumference of a Circle, and thro' the Point B, where it cuts the Line AB, draw thro' the Center C the Diameter BCD, cutting AD in D, through which and the given Point A, draw the Right-Line AD, and that will be a Perpendicular to the Line AB propoſed, that is to ſay, the Angle BAD will be a right one, becauſe 'tis in a Semi-circle.

Fig. 46. This Propoſition ſerves alſo to let fall a Perpendicular from one of the three Angles of a Triangle on the oppoſite Side, or even two at once: Thus if you were to let fall Perpendiculars from the Angles A, B, of the Triangle ABC, on the oppoſite Sides AC, BC, deſcribe upon the third Side AB, the Semi-circle ADEB, and thro' the Points E, D, where the Circumference cuts the Sides AC, BC, draw to the Angles propos'd A, B, the Right-Lines AE, BD, and they will be perpendicular to the Sides BC, AC, by the Property of the Semi-circle.

Fig. 53. I ſhould never have done, if I ſhould endeavour to reckon up all the different Uſes of this Propoſition: I ſhall therefore content my ſelf with ſaying, it is of uſe in Trigonometry, for computing the Table of Sines: in Arithmetick, by Geometry, for ſubſtracting ſimilar Figures; and demonſtrating the following Propoſition, and furniſhing us with an eaſier Method than that in *Prop.* 17. for drawing a Tangent thro' a given Point without the Circumference of a given Circle. Thus if from the Point A, you would draw a Right-Line, that ſhould be a Tangent to the Circle CEB, whoſe Centre is D: Draw from the Centre D, to the Point given A, the Right-Line AD, upon which deſcribe the Semi-circle AED, cutting the Circumference of the given Circle in the Point E, thro' which and the given Point

A,

A, draw the Right-Line AE, and it shall be the Tangent sought by *Prop.* 16. for the Angle AED being in the Semi-circle is a right one.

PROPOSITION XXXII.

THEOREM XXVIII.

A Right-Line cutting the Circumference of a Circle at the Point of Contact, makes two Angles with the Tangent equal to those in the alternate Segments.

AN *Alternate Segment* is that which is on the other Side of the Rectilineal Angle made at the Point of Contact, as ADEA, in regard of the opposite Angle CAD, made by the Line AD, at the Point of Contact A, with the Tangent AC; or the Segment ADFA, in regard of the opposite Angle BAD, form'd by the same Line AD, with the Tangent AB, at the same Point of Contact A. Plate 4. Fig. 47.

I say, first, then that the Angle CAD is equal to the Angle made in the alternate Segment ADEA, for instance to the Angle AED made by the Line ED, with the Diameter AE.

DEMONSTRATION.

Because the Angle ADE is right, by *Prop.* 31. the two other Angles AED, EAD, of the Triangle ADE, are taken together equal to one right, by 32. 1. and consequently equal to the Angle CAE, which is also right by *Prop.* 16. wherefore taking away the common Angle EAD, 'tis evident the single Angle AED, is equal to the Angle CAD. *Which was to be demonstrated.*

I say in the second Place, if you draw thro' the Point F, taken at Discretion in the Arc AFD, the Lines AF, DF, the Angle BAD is equal to the Angle AFD, made in the alternate Segment ADFA.

DEMONSTRATION.

Because in the Quadrilateral Figure AEDF, the Sum of the two opposite Angles E, F, is equal to two right ones, by *Prop.* 22. and consequently equal to the Sum of BAD, CAD, which are also equal to two right ones by *Prop.* 13. 1. taking away the Angles AED, CAD, demonstrated

ſtrated to be equal, 'tis evident the ſingle Angle BAD, is equal to the ſingle Angle F. *Which was to be demonſtrated.*

SCHOLIUM.

Plate 4. Fig. 47. We all along ſuppoſed in both the Demonſtrations, that the Line AD was without the Center G; for if it paſſed through it, as AE does, it would make with the Tangent CB two Right-Angles by *Prop.* 18. and the Angles in the Semi-circles would alſo be right, by *Prop.* 31. Thus the Propoſition is evident.

USE.

Fig. 48. This Propoſition ſerves to demonſtrate *Prop.* 33, and 34. and *Prop.* 10. 4. and that if ſeveral Circles touch one another in the ſame Point, as A, and a Line be drawn thro' it, cutting their Circumferences, as AF, the Arcs of each Circle terminated by that Line, namely AD, AE, AF, are ſimilar Parts of their Circumferences, becauſe all Angles made in the alternate Segments are equal; each being equal to the Angle made by the Right-Line AF and Tangent BC.

PROPOSITION XXXIII.

PROBLEM V.

Fig. 49. *To deſcribe on a given Right-Line a Segment of a Circle, that ſhall contain any given Angle.*

'TIS evident by *Prop.* 31. that if the Angle given be right, you have nothing to do but to deſcribe a Semi-circle on the given Line AB, for that Segment of a Circle will contain a right Angle. But if the given Angle be not right, make on the Extremity B of the given Right-Line AB, the Angle ABC equal to the given one by drawing BC, to which draw the Perpendicular BD, from the Point B, then make on the other Extremity A, the Angle BAE, equal to the Angle ABE, and that will make the Sides AE, BE, of the Triangle ABE, equal by 6. 1. you can therefore deſcribe on the Point E, as a Center thro' the two Extremities A, B, a Circumference of a Circle, and the Segment ABDA ſhall be capable of containing the given Angle, or its equal ABC.

DE-

Plate 4. Fig. 49.

DEMONSTRATION.

Because the Line BC is perpendicular to the Diameter BD, by *constr.* it follows by *Prop.* 18. that 'tis a Tangent to the Circle at the Point B, and by *Prop.* 32. the Segment ABDA can contains an Angle equal to the Angle ABC, equal by Construction to the Angle given. *Which was to be demonstrated.*

USE.

By the help of this Proposition you may find a Point from whence the two unequal Parts of a Line divided into two Parts will appear equal, namely by making on one of the given Lines any kind of Segment of a Circle, and on the other a Segment of a Circle similar to the former; for the Points where the Circumferences of the two Segments intersect, will be that from whence the two Lines proposed being seen under equal Angles, will appear equal.

PROPOSITION XXXIV.

PROBLEM VI.

To cut off a Segment capable of containing any given Angle, from a given Circle.

'TIS evident by *Prop.* 31. that if the Angle given be right, only draw any Diameter in the Circle given, and that will cut off on each Side a Semi-circle, that will contain a Right-Angle: But if the Angle given be not a right one, draw by *Prop.* 16. a Tangent BC to the Point A, taken at Discretion in the Circumference of the given Circle, and draw the Line AD, making the Angle CAD at the Point A, equal to the given one, and it will cut off from the Circle given, the Segment ADEA, that can contain the Angle CAD, and consequently the given Angle, as is evident by *Prop.* 32. Fig. 50.

I

PRO-

PROPOSITION. XXXV.

THEOREM XXIX.

Two Right-Lines crossing one another in a Circle, the Rectangle under the two Parts of the one, is equal to the Rectangle under the two Parts of the other.

Plate 4. Fig. 51. THese two Lines may intersect one another several ways, as in the Center, and then their Parts will be equal, or one passing thro' the Center may bisect the other that does not, and then they will be perpendicular to each other, by *Prop.* 3. or one passing thro' the Centre may cut the other that does not, into two unequal Parts: Or lastly, the two Lines may cut one another without the Centre. I say, in all these Cases the Rectangle under the two Parts of the one, are equal to the Rectangle under the two Parts of the other.

Demonstration of the first Case.

'Tis evident, if the two Lines intersect in the Centre, that their Parts are equal, because each is equal to the Radius of the Circle, consequently their Rectangles are equal, being Squares of the same Radius. *Which was to be demonstrated.*

Plate 4. *Demonstration of the second Case.*

Fig. 51. If one of the two Lines, as AB, pass thro' the Centre, and cutting the other that does not pass thro' the Centre at right Angles, bisects it in the Point E, by 5. 2. you may find that the Rectangle under the Parts AE, BE, together with the Square of the intermediate Part EF, is equal to the Square of FB, or FC, or by 47. 1. to the two Squares EF, FC, wherefore substracting the common Square EF, you will find the single Rectangle under the Parts AE, BE, is equal to the Square EC alone, that is to say to the Rectangle under the Parts EC, ED. *Which was to be demonstrated.*

Demonstration of the third Case.

Fig. 52. If one of the two Lines AB, CD, insecting one another,

ther, without the Centre in the Point E, as AB pass thro' the Centre F of the Circle, and is not perpendicular to the other CD, let fall FG perpendicular to the other CD, from the Center F, and it will bisect it in the Point G, by *Prop.* 3. and draw the Radius FC, then by 5. 2. the Rectangle under the Parts CE, DE, together with the Square of the intermediate Part EG, is equal to the Square of the half CG; wherefore adding the Square FG, the Rectangle under the Lines CE, DE, together with the Sum of the Squares FG, EG, or by 47. 1. with single Square FE, is equal to the Squares CG, FG, or by 47. 1. to the single Square FC or FB, or by 5. 2. to the Rectangle under the Lines AE, BE, and to the Square of the intermediate Part FE, which taken from each Side, leaves the single Rectangle under the Parts CE, DE, equal to the single Rectangle under the Parts, AE, BE. *Which was to be demonstrated.* Plate 4. Fig. 53.

Demonstration of the fourth Case.

Lastly, If neither of the two Lines CD, HI, intersecting one another in a Point E without the Circle, pass thro' the Centre F, you may easily demonstrate that the Rectangle under the Parts CE, DE, is equal to the Rectangle under the Parts EH, EI, because, drawing the Diameter AB thro' the Point E, 'tis evident from the preceding Case, that each of these two Rectangles is equal to the Rectangles under the Parts AE, BE, and consequently equal to one another. *Which was to be demonstrated.*

USE.

This Proposition serves to demonstrate several Theorems in Trigonometry, and to find a Mean proportional betwen two given Lines; for instance, AE, BE, for having placed them in a Right-Line, describe the Semicircle ABC, upon their Sum AB, and erect the Perpendicular EC, upon the Point E, of the Line AB, and that shall be the mean proportional sought, as has been demonstrated in *Prop.* 13. 6. you may also find a third Proportional to two, or a fourth to three given Lines. Fig. 51.

Plate 4. Fig. 53.

PROPOSITION XXXVI.

THEOREM XXX.

A Tangent and Secant being drawn from the same Point taken at Pleasure without the Circle; the Square of the Tangent will be equal to the Rectangle under the whole Secant, and its external Part.

I Say, first, the Square of the Tangent AE, is equal to the Rectangle under the whole Secant AB, that passes thro' the Center D, and its external Part AC.

DEMONSTRATION.

Draw the Radius DE thro' the Centre D and Point of Contact, and by *Prop.* 18. the Triangle ADE is right-angled in E, and by 6. 2. the Rectangles under the Lines AB, AC, with the Square CD or DE, is equal to the Square of the Line AD, that is to say, to the two Squares AE, DE, by 47. 1. wherefore taking away the common Square DE, 'tis plain the Rectangle under the Lines AB, AC, is equal to the single Square AE. *Which was to be demonstrated.*

Fig. 54. I say, in the second Place, the Square of the Tangent AE, is equal to the Rectangle under the Line AB, that does not pass thro' the Centre and its external Part AC.

PREPARATION.

Draw as before the Radius DE, and that will be perpendicular to the Tangent AE, by *Prop.* 18. Draw also the Radius DC, and let fall from the Centre D, the Line DG perpendicular to the Line AB, and it will bisect it in G. Lastly, Join the Right-Line AD.

DEMONSTRATION.

Because the Rectangle under the Lines AB, AC, with the Square CG, is equal to the Square AG, by 6. 2. adding to each Side the Square DG, the Rectangle under the Lines AB, AC, together with the Sum of the two Squares CG, DG, that is to say, by 47. 1. with the single

single Square CD or DE, is equal to the Sum of the Squares AG, DG, or by 47. 1. to the single Square AD, or the two Squares AE, DE; wherefore take away the common Square DE, and you will find the single Rectangle under the Lines AB, AC, equal to the single Square AE. *Which was to be demonstrated.* Plate 4. Fig. 54.

COROLLARY I.

From hence it follows that drawing a Right-Line, as AH, from the same Point A, the Rectangle under that Line AH, and its Part AI, is equal to the Rectangle under the whole Line AB, and its external Part AC, because each of these Rectangles is equal to the same Square, namely, the Square of the Tangent AE.

COROLLARY II.

From hence also it follows, that if you draw another Tangent AF, from the same Point A, that Tangent AF, will be equal to the first AE, because the Square of each is equal to the Rectangle under the Lines AB, AC, or the Rectangle under the Lines AH, AI.

USE.

We shall make use of this Proposition in Trigonometry, to find, otherwise and easier than by *Prop.* 15. 2. the Segments of the Base of a Triangle made by a Perpendicular falling from the Angle opposite to the Base, which serves to find the Area of the Triangle, as also to find the Angle, as shall be seen in Trigonometry. This Proposition serves also to demonstrate the following one, which is its converse. Fig. 54.

PROPOSITION XXXVII.

THEOREM XXXI.

If the Rectangle under the Secant, and its external Part, be equal to the Square of a Line meeting the Circumference of a Circle, that Line is a Tangent.

I Say, if the Rectangle under the Secant AB, and its external Part AC, be equal to the Square of the Line AE, meeting in E the Circumference of the Circle EFH whose Fig. 54.

Plate 4. Fig. 54. whose Centre is D, the Right-Line AE is a Tangent to the Circle in that Point E.

DEMONSTRATION.

Draw the Right-Line AD, Tangent AF, and Radij DE, DF, by *Prop.* 36. the Square of the Tangent AF is equal to the Rectangle of the Lines AB, AC; and since AE Square is suppos'd equal to the same Rectangle, it follows that the Line AE, AF are equal, and by 8. 1. the Angle E is equal to the Angle F, which being right by *Prop.* 18. the Angle E will be right, and by *Prop.* 16. the Line AE will be a Tangent in the Point E. *Which was to be demonstrated.*

USE.

This Proposition serves to demonstrate *Prop.* 10. 4. and that but two Tangents can be drawn from the same Point taken at pleasure without the Circle, because by this and the last, the two Tangents AE, AF, being equal, if a third could be drawn as AI, it would also be equal to the two foregoing AE, AF, and so more than two equal Lines could be drawn from the same Point to the Convex Circumference of a Circle, contrary to *Prop.* 8. There are other Uses but less considerable, which I omit, that I may come to the following Book.

THE

The FOURTH BOOK of

EUCLID's ELEMENTS.

EUclid having explained the principal Properties of the Circle, gives us here several Problems for inscribing and circumscribing regular Polygons, which is of vast use in the Fortification of regular Places, and making Tables of Sines in Trigonometry, and Squaring the Circle in Geometry, to which you may approach, as near as you please, by inscribed and circumscribed Polygons, and for explaining the different Aspects of Planets in Astrology, that take their Names from Polygons determining their Distances, by the relation to that Part which this Distance is of the whole Circumference of a great Circle, that passes thro' the Centers of the Planets.

DEFINITIONS.

I.

A Rectilineal Figure is said to be inscribed in another Rectilineal Figure, when the Vertex of each of its Angles touches one of the Sides of the Figure that 'tis inscribed in. *Thus the Figure EFGH, is inscribed in the Figure ABCD.* Fig 1.

II.

A Rectilineal Figure is circumscribed about another Rectilineal Figure, when each of its Sides passes thro' the Vertex of one of the Angles of the Figure about which 'tis circumscribed. *Thus the Figure ABCD is circumscribed about the Figure EFGH.*

These

These two Definitions are of no use in what we have to say, because this Book treats only of Rectilineal Figures inscrib'd or circumscrib'd about a Circle. But because the Commentators have not omitted them, and they may be of use in other Cases, we have not neglected them.

III.

A Rectilineal Figure is said to be inscribed in a Circle, when the Vertex of each of its Angles touches the Circumference of the Circle 'tis inscribed in. *Thus the Triangle* ABC is inscribed in the Circle ABFEC, tho' the Triangle DEF is not, because the Vertex *of the Angle EDF does not touch the Circumference.*

Fig. 3.

IV.

Fig. 6.

A Rectilineal Figure is said to be circumscribed about a Circle, when each of its Sides touches the Circumference of the Circle it is circumscribed about. *Thus the Triangle ABC is circumscribed about the Circle EFG, because its Sides touch the Circumference in the Points E, F, G.*

V.

Fig. 5.

A Circle is said to be inscribed in a Rectilineal Figure, when the Circumference touches each of the Sides of the Figure 'tis inscribed in. *Thus the Circle DEF is inscribed in the Triangle IKL, because its Circumference touches its Sides in the Points D, E, F.*

VI.

Fig. 3.

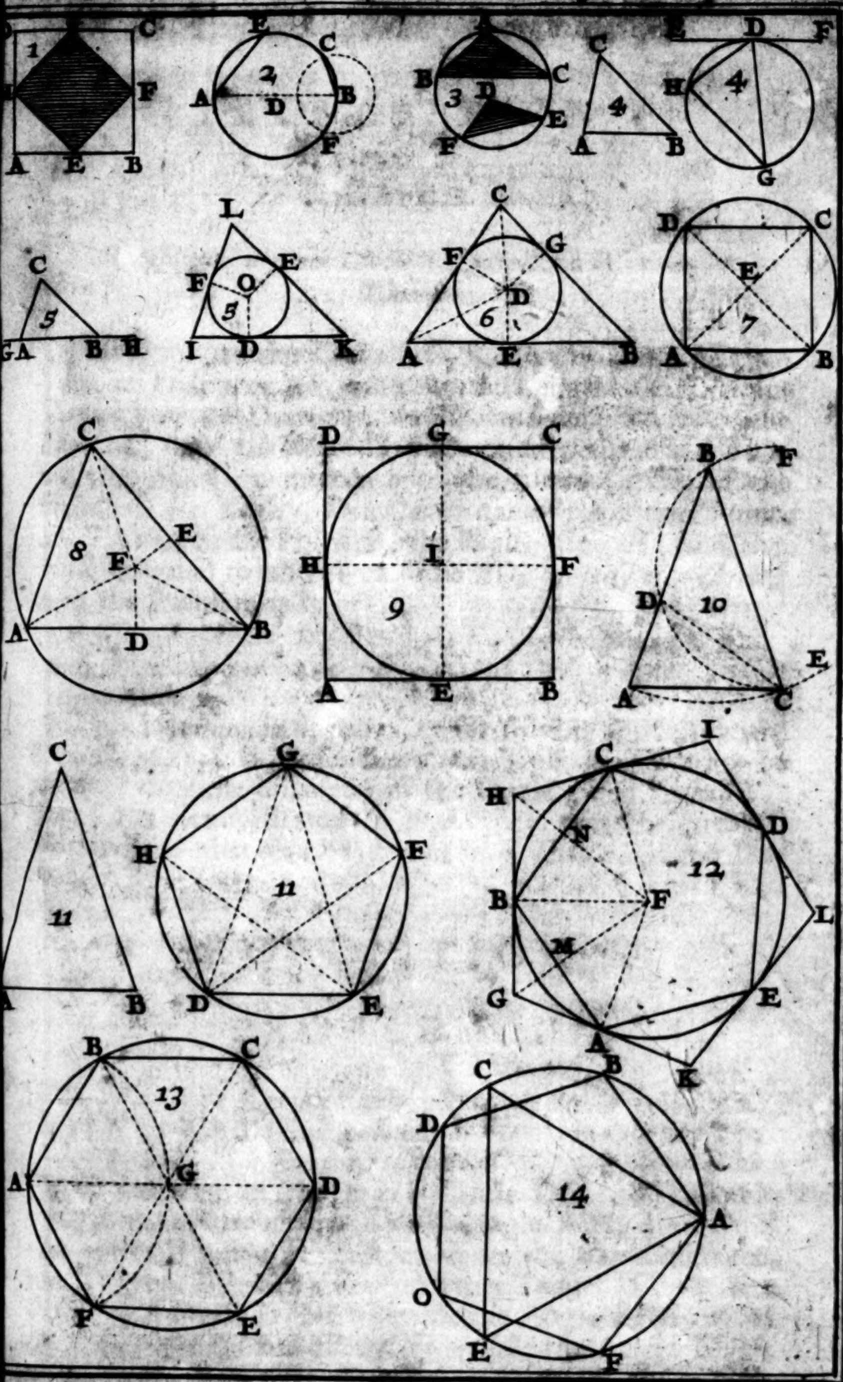
A Circle is circumscribed about a Rectilineal Figure, when its Circumference passes thro' the Vertex of each Angle of the Figure it is said to be circumscribed about. *Thus the Circle ABFEC is circumscrib'd about the Triangle ABC, because its Circumference passes thro' the Vertices of the Triangle A, B, C,*

VII.

Fig. 2.

A Right-Line applied to a Circle, is that whose two Extremities touch the Circumference of the Circle to which it is applied, *as AE.*

PRO-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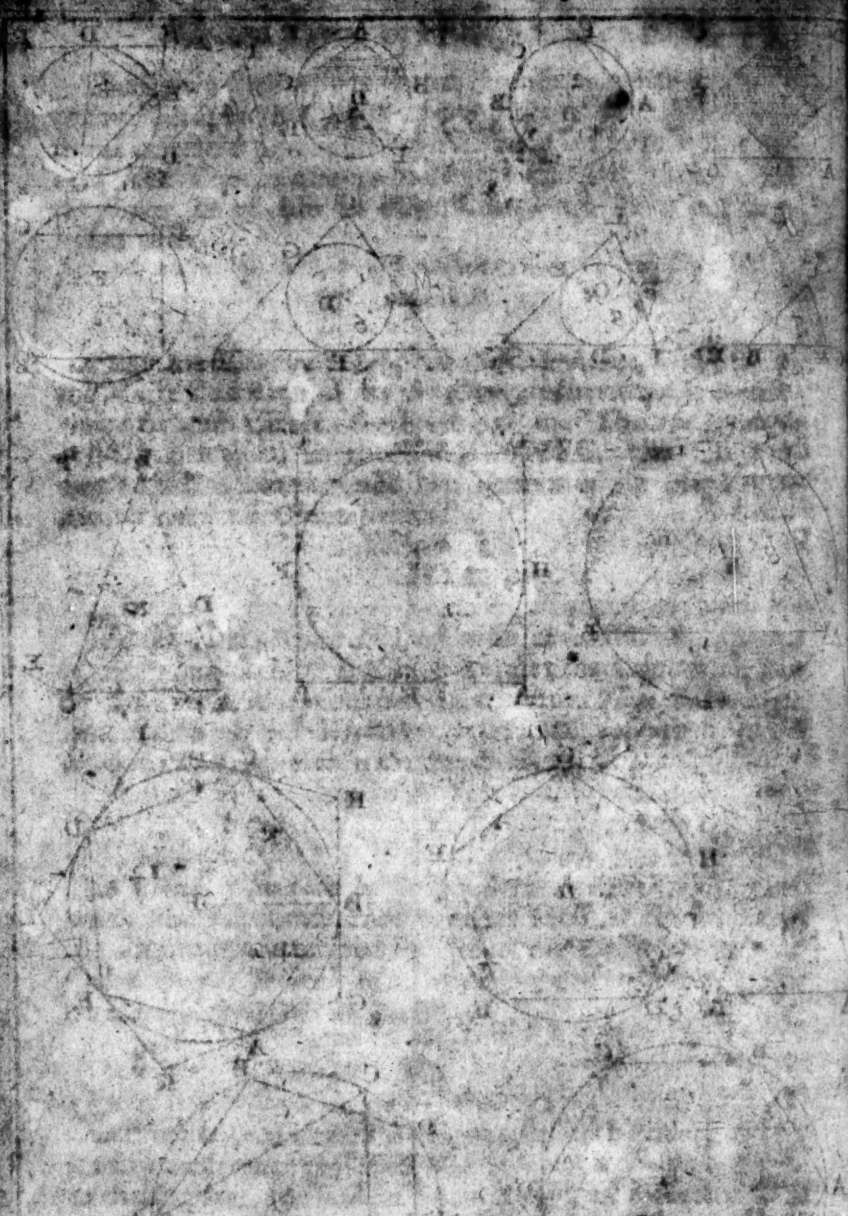


PROPOSITION I.

PROBLEM I.

To apply to a given Circle a Right-Line less than its Diameter.

TO apply to the Circle AECB, a Right-Line less than its Diameter AB, mark out the Length of that Right-Line upon the Diameter, as BD, and describe upon the Point B, thro' the Point D, a Circumference of a Circle, cutting the Circumference of the given Circle in the Points C, F. Lastly, Draw thro' one of these two Points F, C, as C, to the Point B, the Right-Line BC, and that will be equal to the given Line BD, *by Def. of a Circle*; and the Problem is resolv'd. Fig. 1.

USE.

This Proposition is necessary for solving the following Problems, and supposes the given Right-Line not to be greater than the Diameter of the Circle given, because it has been demonstrated in *Prop.* 15. 3. that the greatest Right-Line that can be drawn in a Circle, is the Diameter.

PROPOSITION II.

PROBLEM II.

To inscribe in a given Circle a Triangle Equiangular to a given one.

TO inscribe in the given Circle DGH, a Triangle Equiangular to the given Triangle ABC, draw thro' the Point D taken at Discretion in the Circumference, the Tangent EF, and make with that Tangent EF, at the Point of Contact D, on one side the Angle FDG, equal to the Angle A, and on the other side the Angle EDH, equal to the Angle B. Lastly, Join the Right-Line GH, and the Triangle DGH, will be equiangular to the given one ABC, so that the Angle G will be equal to the Angle B, and the Angle H to the Angle A. Fig. 2.

Fig. 4.

DEMONSTRATION.

Because by 32. 3. the Angle FDG, or A, is equal to the Angle H of the alternate Segment DHGD, and, in like manner the Angle EDH, or B, is equal to the Angle G of the alternate Segment GDHG, it follows *by* 32. 1. that the Third Angle GDH, is equal to the Third Angle C, and thus the Triangle DGH is equiangular to the given Triangle ABC. *Which was to be demonstrated.*

USE.

This Proposition serves to inscribe a regular Pentagon in a given Circle, as you will find in *Prop.* 11. or a regular Pentedecagon, as shall be shown in *Prop.* 16.

PROPOSITION III.

PROBLEM III.

To circumscribe about a given Circle a Triangle equiangular to a given one.

Fig. 5.

TO circumscribe about the given Circle DEF, whose Center is O, a Triangle equiangular to the given one ABC, draw any Radius OD, and producing the Base AB of the given Triangle ABC, towards G, and H, make at the Center O, with the Radius OD, on one side the Angle DOE equal to the external Angle CBH, on the other side the Angle DOF equal to the other external Angle CAG. Lastly, draw thro' the Points E, F, D, the Tangents IK, KL, LI, and they will make the Triangle IKL equiangular to that propos'd ABC, and circumscrib'd about the given Circle DEF.

DEMONSRATION.

Since the three sides of the Triangle IKL touch the Circle DEF, *by Constr.* 'tis evident *by Def.* 4. the Triangle IKL is circumscrib'd, and *by* 16. 3. the three Angles D, E, F, are Right; and because *by* 32. 1. the four Angles of the Trapezium KDOE, are taken together equal to four Right, and the two E, D, are Right, it follows also that the two others DOE, and K, are taken together equal to two Right ones, and consequently to the

two

two HBC, ABC, that are also equal to two Right ones, by 13. 1. and because the Angle DOE is equal to the Angle HBC, by *Constr.* the Angle K must necessarily be equal to the Angle ABC. After the same manner the Angle I may be demonstrated to be equal to the Angle BAC. Whence 'tis easy to conclude, by 32. 1. that the Triangle IKL is equiangular to the Triangle ABC. *Which was to be demonstrated.* Fig. 5.

PROPOSITION IV.

PROBLEM IV.

To inscribe a Circle in a given Triangle.

TO inscribe a Circle in the given Triangle ABC, bisect its two Angles, as A and C, by the Right-Lines AD, CD, and let fall from the Point D, where they intersect, the Perpendiculars DE, DF, DG to the three sides of the Triangle propos'd ABC, and they will be equal; so that a Circle describ'd upon the Center D, thro' the Point E, will pass thro' the Points F, G. Fig. 6.

DEMONSTRATION.

Because the Angles E, F, are equal, being Right, by *Constr.* and the Line AD bisects the Angle BAC, the two Triangles ADE, ADF, will be equal, by 26. 1. and the side DE will be equal to the side DF. After the same manner the two right-angled Triangles CDF, CDG, may be demonstrated to be equal, and consequently the side DF equal to the side DG. Whence it follows, that the three Perpendiculars DE, DF, DG, are equal, and that a Circle may be describ'd upon the Center D, thro' the three Points E, F, G; and since the Angles made at the three Points E, F, G, are Right, the sides of the Triangle ABC, will be Tangents to the Circumference of the Circle, consequently the Circle is inscrib'd in the Triangle. *Which was to be demonstrated.*

USE.

This Proposition serves to demonstrate, that three Right-Lines bisecting the Angles of a Triangle, meet in the same Point within the Triangle, because the Center of the Circle that may be inscrib'd in that Triangle, is in each of those Lines.

PROPOSITION V.

PROBLEM V.

To circumscribe a Circle about a given Triangle.

Fig. 8. TO circumscribe a Circle about the given Triangle ABC, bisect two of its sides, as AB, BC, in the Points D, E, from whence let fall the Perpendicular DF, EF, and the Point F, where they intersect, will be the Centre of the Circle sought, so that the three Lines FA, the FB, FC, are equal.

DEMONSTRATION.

You know by 4. 1. the two right-angled Triangles ADF, BDF, are equal, and consequently the two Lines AF, BF, are equal. After the same manner, you may know, that the two Lines BF, CF, are also equal. Whence it follows, that the three Lines AF, BF, CF, are equal, and consequently that upon the Point F, as a Center, a Circle may be describ'd, whose Circumference will pass thro' the Points A, B, C, which therefore will be circumscrib'd about the Triangle ABC. *Which was to be demonstrated.*

USE.

This Proposition serves to demonstrate that the three Perpendiculars, erected upon the middle of the sides of a Triangle, intersect in the same Point, because each passes thro' the Center of the Circle that may be circumscrib'd.

PRO-

PROPOSITION VI.

PROBLEM VI.

To inscribe a Square in a given Circle.

TO inscribe a Square in the given Circle ABCD, draw thro' its Center E, any Diameter as AC, and another as BD perpendicular to it, join the Right-Lines AB, AD, BC, CD, and the Rectilineal Figure ABCD will be a Square. Fig 7.

DEMONSTRATION.

The four Angles of the Rectilineal Figure ABCD, are Right, by 31. 3. because they are in Semi-circles; and its four Sides are equal, because they are the Hypotenuses of the four right-angled Triangles AEB, BEC, CED, AED, that are equal by 4. 1. Consequently the Rectilineal Figure ABCD is a Square. *Which was to be demonstrated.*

PROPOSITION VII.

PROBLEM VII.

To circumscribe a Square about a given Circle.

TO circumscribe a Square about the given Circle EFGH, whose Center is I, draw at Pleasure the two perpendicular Diameters EG, FH, and draw thro' the four Points E, F, G, H, the Tangents AB, BC, CD, AD, and they will make the Square ABCD, which will circumscribe the Circle EFGH. Fig. 9.

DEMONSTRATION,

'Tis evident the Figure ABCD is circumscrib'd about the Circle EFGH, because all its Sides touch the Circumference, by *Constr.* 'Tis evident also that the same Figure ABCD is a Square, these Angles made at the four Points E, F, G, H, being Right, and consequently the four Squares AI, BI, CI, DI, that compose the Figure ABCD, are equal, *&c.*

PROPOSITION VIII.

PROBLEM VIII.

To inscribe a Circle in the given Square.

Fig. 8. TO inscribe a Circle in a given Square ABCD, bisect each of the Sides in the Points E, F, G, H, and join the Right-Lines EG, FH, and the Point of Intersection I, will be the Center of the Circle sought, which may consequently be drawn thro' the four Points E, F, G, H, because the four Lines IE, IF, IG, IH, are equal.

DEMONSTRATION.

Because the Lines AH, BF, are equal and parallel, the Lines AB, FH, will be also equal and parallel, by 33. 1. And so the Figure AF will be a Parallelogram; by the same way you may find, that the Figures CE, CH, DF, are Parallelograms equal to the first AF: and since they are Rectangles, and bisected by the Lines, that proceed from the Point I, it follows that their Halves AI, BI, CI, DI, are equal Squares, and consequently the Lines IE, IF, IG, IH, are equal. *Which was to be demonstrated.*

PROPOSITION IX.

PROBLEM IX.

To circumscribe a Circle about a given Square.

Fig. 7. TO circumscribe a Circle about the Square ABCD, draw the two Diagonals AC, BD, and the Point E, of their Section, will be the Center of the Circle sought: so that the four Lines EA, EB, EC, ED, are equal.

DEMONSTRATION.

Because all the acute Angles of the four Triangles AEB, AED, CEB, CED, by 4. 2. are Semi-right, and

and consequently equal, as well as the Sides AB, BC, CD, AD, because they are the Sides of the Square ABCD, these four Triangles, by 26. 1. will be equal, and consequently their Sides EA, EB, EC, ED. So that a Circle may be describ'd upon the Point E, as a Center, thro' the Points A, B, C, D. *Which was to be demonstrated.* Fig. 7

PROPOSITION X.

PROBLEM X.

To make an Isosceles Triangle, where each of the two Angles at the Base shall be double the third.

To make the Isosceles Triangle ABC, in which each of the two Angles at the Base A and C, are double the third Angle B, draw the Line AB what length you please, and divide it at the Point D, *by* 11. 2. so that the Square of BD be equal to the Rectangle under AB, AD: And having describ'd the Arc ACE, upon the Point B, thro' the Point A, apply to it, by *Prop.* 1. the Right-Line AC equal to BD, and join the Right-Line BC, then will ABC be the Triangle sought. Fig. 10.

DEMONSTRATION.

'Tis evident the Triangle ABC is Isosceles, that is to say, the two Legs BA, BC, are equal, for the Point B, by *Constr.* is the Center of the Arc ACE. Whence it follows, by 5. 1. that the Angles A and C are equal: What remains to be demonstrated is, that each is double the Angle B, which will be done by drawing the Right-Line CD, and a Circumference thro' the three Points B, C, D; and then reason thus.

Because the Rectangle under the whole Line AB, and its Part AD, is, by *Constr.* equal to the Square of the other Part BD, or AC, its equal, the Line AC will be a Tangent in the Point C to the Circumference FBDC, *by* 37. 3. and 32. 3. the Angle ACD will be equal to the Angle B; and since, by 32. 1. the external Angle ADC is equal

 to

Fig. 19. to the Sum of the two internal and opposite B, BCD, or ACD, BCD, that is to say, to the whole Angle BCA, or the Angle A, it follows, by 6. 1. that the Line AC, or BD, is equal to the Line CD, and, by 5. 1. the Angle B, or ACD, is equal to the Angle BCD, and consequently the whole BCA, or the Angle A, its Equal, is double the Angle C. *Which was to be demonstrated.*

USE.

This is subservient to the following one, and serves for inscribing a regular Decagon in a Circle, because the Line AC, apply'd in the Circle, whose Radius is AB, is the Side of a Decagon that may be inscrib'd in it, the Angle B being 36 Degrees, the 10th part of the whole Circle, or 360 Degrees. Thus you see the Radius AB, which is the Side of a Hexagon, as shall be demonstrated in *Prop.* 15. being *by* 11. 2. cut in extreme and mean Proportion at the Point D, the greater Part BD is equal to the Side of the Decagon, and you will find by the next Proposition, that the greater Part BD, is the Side of a regular Pentagon, that may be inscrib'd in a Circle circumscrib'd about an Isosceles Triangle ABC.

PROPOSITION XI.

PROBLEM XI.

To inscribe a regular Pentagon in a Circle.

Fig. 18. TO inscribe a regular Pentagon in the given Circle DEFGH, make, by *Prop.* 10. the Isosceles Triangle ABC, in which each of its two Legs at the Base A, B, shall be double the third C, and, by *Prop.* 2. inscribe in the given Circle the Triangle DEG equiangular to the Triangle ABC, and so the two Angles at the Base GDE, GED, will be each double the third Angle DGE. Wherefore bisect each of these two Angles GDE, GED, by the Right-Lines DF, EH, and join the Points E, F, G, H, D, by Right-Lines, and the Figure DEFGH will be a regular Pentagon, that is to say, equilateral and equiangular.

DEMON-

DEMONSTRATION.

Because the Angles DGE, EDF, FDG, GEH, DEH, are halves of the Angle GDE, or GED its equal, by *const.* they will be equal to one another, and by 26. 3. the Arcs DE, EF, FG, GH, DH, on which they insist, will also be equal, consequently by 29. 3. the Lines DE, EF, FG, GH, DH, are also equal. Thus you see the Pentagon DEFGH, is equilateral and equiangular, because each of its Angles insist upon three equal Arcs. *Which was to be demonstrated.* Fig. 11.

USE.

This Proposition serves not only for Citadels, that are usually made of five Bastions, but for resolving the next and the 16th Proposition, and besides opens the way for uneven Polygons: For 'tis evident that to inscribe for instance an Heptagon in a given Circle, you must know how to make an Isosceles Triangle in which each of the two Angles at the Base is triple the Third: But it being a solid Problem, *Euclid* has not resolved it.

PROPOSITION XII.

PROBLEM XII.

To circumscribe a regular Pentagon about a given Circle.

TO circumscribe a regular Pentagon about a given Circle ABCDE, whose Centre is F, you must inscribe by *Prop.* 11. the regular Pentagon ABCDE, and draw Tangents by 17. 3. thro' the Points A, B, C, D, E, and you will have the Pentagon sought. Fig. 12.

DEMONSTRATION.

Drawing from the Center F, the Lines FA, FG, FB, FH, FC, you will find by 8. 1. the Triangles FGA, FGB, are equal, the Side FG being common, and the two Radij FA, FB, equal by *Def. of a Circle*, and the two Tangents GA, GB, equal by 36. 3. consequently the Angles

Fig. 12. Angles AFG, BFG, will be equal as well as FGA, FGB and by the same method you may find that the two Angles BFH, CFH, are equal, as well as BHF, CHF; and because the whole Angle AFB is equal to the whole Angle BFC, by 27. 1. the Arcs AB, BC, being equal by *const.* their halves BFG, BFH, will also be equal. From whence 'tis easy to conclude that the four Triangles AFG, BFG, BFH, CFH, are also equal, and may be demonstrated after the same manner, drawing other Right-Lines from the Center F, thro' the Points I, D, L, E, K, and consequently the Pentagon GHILK is equilateral and equiangular. *Which was to be done and demonstrated.*

PROPOSITION XIII.

PROBLEM XIII.

To inscribe a Circle in a Regular Pentagon.

Fig. 12. TO inscribe a Circle in the Regular Pentagon GHILK, do as you did in the Case of a Triangle, that is to say, bisect two of its Angles, as G, H, by the Right-Lines GF, HF, and the Point F of their Section will be the Centre of the Circle sought, so that letting fall from the Centre F the Perpendiculars FA, FB, FC, to the Sides GK, GH, HI, *&c.* they will be equal.

DEMONSTRATION.

Because the Angle FGB is equal to the Angle FGA, by *const.* and the Side FG, common to the two Triangles FAG, FBG, right-angled in A and B, by *constr.* they they will be equal by 26. 1. and the Perpendicular FA, will be equal to the Perpendicular FB, and consequently the three Perpendiculars FA, FB, FC, and all the rest, that can be let fall from the Point F, on the Sides of the Pentagon propos'd, are equal to one another. Thus you have found the Point F, on which a Circle may be

describ'd,

describ'd, whose Circumference will touch the Sides of the regular Pentagon GHILK. *Which was to be demonstrated.* Fig. 12.

PROPOSITION XIV.

PROBLEM XIV.

To circumscribe a Circle about a Regular Pentagon.

TO circumscribe a Circle about the Regular Pentagon, ABCDE, do as in the Case of a Triangle, that is to say, bisect two of its Sides, as AB, BC, at the Points M, N, and erect the Perpendiculars MF, NF, from the Points M, N, and the Point F of their Section will be the Centre of the Circle, so that if you draw from the Centre F, to the Angles of the Pentagon proposed, the Right-Lines FA, FB, FC, *&c.* they will all be equal. Fig. 12.

DEMONSTRATION.

Because the Line AM is equal to the Line BM, by *const.* and the Side FM common to the two Triangles FMA, FMB, right-angled in M, by *const.* these two right-angled Triangles FMA, FMB will be equal by 4. 1. and their Hypotenuses also, FA, FB. After the same manner the Hypotenuse FC of the right-angled Triangle FNC, may be demonstrated to be equal to the Hypotenuse FB of the right-angled Triangle FMB, and consequently the three Lines FA, FB, FC, and all others, that can be drawn from the Centre F, thro' the Angles of the Pentagon propos'd, are equal to one another. And so the Point F is found, upon which a Circle may be described, whose Circumference will pass thro' all the Angles of the given Pentagon ABCDE. *Which was to be demonstrated.*

SCO-

SCHOLIUM.

The three foregoing Problems applied to a Regular Pentagon, may be applied after the same manner, to any other Regular Polygon, and for that Reason *Euclid* speaks no more of it in what follows.

PROPOSITION XV.

PROBLEM XV.

To inscribe a Regular Hexagon in a Circle.

Fig. 13. TO inscribe a Regular Hexagon in the Circle ABCDEF, whose Centre is G, draw any Diameter as AD, and describe the Arc BGF, from its Extremity A, thro' the Centre G, cutting the Circumference of the given Circle in the Points B, F, thro' which draw the Diameters BE, FC, and then the Lines AB, BC, CD, DE, EF, AF, and the Figure ABCDEF will be a Regular Hexagon, that is to say equilateral and equiangular.

DEMONSTRATION.

Because each of the two Triangles AFG, ABG, is equilateral, 'tis also equiangular by 5. 1. and each of the two Angles AGF, AGB, is a third of two right ones, by 32. 1. as well as their equals, and opposite at the Vertex CGD, DGE, by 15. 1. Whence 'tis easy to conclude, that each of the two other equal Angles BGC, EGF, is also a third of two right ones, because the three AGB, BGC, CGD, taken together are equal to two right ones, and so the Angles at the Centre being equal, the Hexagon ABCDEF will be a regular one. *Which was to be effected and demonstrated.*

USE.

This Proposition serves to discover to us, that the Side of an Hexagon, inscrib'd in a Circle, is equal to the Radius or Semi-Diameter of the same Circle, and that

that furnishes us with a Method of dividing the Circumference of a Circle into six equal Parts, by applying the Radius six times to the Circle; and 'tis with this they generally begin in dividing the Circumference of a Circle into 360 equal Parts or Degrees, as has been seen in *Prob.* 7. *Introd.*

You see also that an equilateral Triangle may easily be inscribed in a Circle by this Proposition, for having divided its Circumference into six equal Parts, as has been taught, join every other Point by Right-Lines, and those three Lines will form an equilateral Triangle.

The use of the Sector in respect to the Line of Polygons, is founded on this Proposition, that shews us also that the Sine of an Arc of 30 Degrees is equal to half the Radius, and the making Tables of Sines is generally begun with this Problem, as shall be seen in the *Treatise of Trigonometry*.

PROPOSITION XVI.

PROBLEM XVI.

To inscribe a Regular Pentedecagon in a Circle.

TO inscribe in the Circle ABCDEF, a Regular *Pentedecagon*, or Figure of fifteen Sides, inscribe by *Prop.* 2. or 15. the equilateral Triangle ACE, and by *Prop.* 11. the regular Pentagon ABDOF, so that the Triangle and Pentagon may have one of their Angles at the same Point A; then the Arc CD will be a fifteenth Part of the Circumference. Fig. 14.

DEMONSTRATION.

Imagine the Circumference to be divided into fifteen equal Parts, then the Arc AB or BD, will contain three, because the Arcs are each a fifth Part of the Circumference by *const.* The Arc AC also will contain five, because 'tis a third Part of the Circumference by *const.* Whence 'tis easy to conclude that the Arc BC contains two, consequently the Arc CD one, for substracting three, that are in AB from five that are in AC, and there

Fig. 14. there will remain two for BC, and substracting two that are in BC, from three that are in BD, there will remain one for CD. *Which was to be effected and demonstrated.*

USE.

This Proposition opens the way to other uneven Polygons, for as multiplying 3 by 5, the Product 15, shews that a Polygon of 15 Sides may be form'd by the help of a regular Figure of 3 and 5 Sides: So multiplying 3 for instance by 7, the Product 21 shews that you may describe a Polygon of 21 Sides by the means of a regular Figure of 3 and 7 Sides.

The FIFTH BOOK of

EUCLID's ELEMENTS.

EUclid in this Book treats of Ratios and Proportions, that he may compleat the Doctrine of Planes in the sixth Book, which he treated of singly in the four preceding Books.

As this Book is the Foundation of the sixth and following Books, so 'tis the Foundation of the principal Parts of *Mathematicks*, where Proportions can't be passed over, by reason of the Comparison one is continually obliged to make of some Quantities with others: And 'tis also absolutely necessary for the understanding of all Mathematical Treatises demonstrated by Proportions; for in *Practical Geometry*, for instance, accessible and inaccessible Lines in surveying are measur'd and found by Reasonings depending upon Proportions; *Arithmetic* contains the *Rule of Three*, call'd the *Rule of Proportion*, because perform'd by Proportions: *Astronomy* compares the different Magnitudes of the Planets, and their Orbs, and different Distances from the Earth, or Sun. *Statics* considers the Proportions of Weights; and *Musick* applies them to Sounds. So that you may assure your self, that you can draw no certain Conclusion in *Mathematicks* without the Knowledge of *Proportions*.

DEFI-

DEFINITIONS.

I.

A Part is a less Quantity compar'd with a greater that it exactly measures. *Thus a Line of two Feet is a Part of a Line of six Feet, for it exactly measures it by* 3, *that is, it is contained three times without a remainder.*

Thus *Euclid* defines a Part, commonly call'd an *Aliquot Part*, to distinguish it from what they call an *Aliquant Part*, that does not measure the whole exactly; as a Line of two Feet does not in regard of 5 Feet, being contain'd twice and 1 remaining, and so is as an aliquant Part of 5 Feet.

By a *Whole* is understood a greater Quantity in relation to a less, whether it actually contains it, or does not; and by a *Part* in general, a less Quantity in regard of a greater, whether it measures it or no, as when we say, *The Whole is greater than its Part.*

An Aliquot Part takes its Name and Denomination from the Number of equal Parts a Quantity is divided into, that is to say, the Number of times 'tis contained in that Quantity or *Whole*. Thus an Aliquot Part that is contain'd twice in any Quantity is call'd an half, and is writ thus $\frac{1}{2}$; and that which is contain'd thrice, is call'd a third, and express'd thus, $\frac{1}{3}$, *&c.*

An Aliquant Part has sometimes aliquot Parts, that measure the Quantity 'tis a Part of; thus for instance 6, which is an aliquant Part of 8, has for its aliquot Part 2, which is a Quarter of 8, of which consequently 6 is three Quarters, since 6 contains 2 three times, and is expressed thus, $\frac{3}{4}$.

Parts, whether aliquant or aliquot, are call'd *Fractions*, in respect of the whole of which they are Parts; and when express'd by Numbers, as we shall hereafter do; the upper Number is call'd, *The Numerator of the Fraction*, and the under, *The Denominator of the same Fraction.* Thus in this Fraction $\frac{2}{5}$ signifying two fifths, the Numerator is 2, and the Denominator 5.

II. A

II.

A Quantity is a Multiple of another, that contains that other a certain Number of Times exactly, that is to say without any Remainder. *Thus a Line of six Feet is the multiple of a Line of two Feet, because it contains it three times exactly.*

'Tis evident the Multiple is greater than that Quantity whose Multiple it is said to be, it being an aliquot Part of it, and call'd a *Submultiple*, in respect of its Multiple, that takes its Name and Denomination from the Number of Times, it contains its Submultiple. Thus a Line of 6 Feet is call'd the *Triple* of a Line of 2 Feet, because it contains it 3 times exactly; but a Line of two Feet is call'd the *Subtriple* of a Line of 6 Feet, because it is contain'd in it three times precisely.

III.

Equimultiples of several Quantities, are Quantities that contain equally, or an equal Number of Times, or as many Times, the Quantities whose Equimultiples they are said to be, that is to say, their aliquot Parts, or Submultiples, which consequently measure their Equimultiples equally. *Thus because a Line of 12 Feet contains a Line of 2 Feet, as many Times as a Line of 30 Feet does a Line of 5 Feet, the two Lines of 12 Feet and 30 Feet are Equimultiples of the Lines of 2 Feet and 5 Feet.*

Thus *Euclid* defines Equimultiples, but we shall call more generally *Equimultiples of several Quantities*, such as contain the Quantities whose Equimultiples they are, an equal Number of Times, whether that Number be an Integer or Fraction, or Integer and Fractions, provided they be similar Parts.

Thus we know that 5 and 10 are Equimultiples, of 2 and 4, because 5 contains 2 twice and one over, which is half two, and in like manner 10 contains 4 twice, and two over, which are half 4.

'Tis in this Sense we would be understood to speak, when we say two Quantities for instance contain or are contained in two others, an equal Number of Times, each of its own.

By similar Parts of several Quantities, whether aliquot or aliquant, we understand such as are contained an equal number of times by them. Thus 9 and 15 are similar Parts of 12 and 20, because 9 is three quarters of 12, as well as 15 of 20.

When any two Quantities are multiplied by the same Quantity, the two Quantities produced by that Multiplication are Equimultiples of the two former, which consequently are similar Parts of the two latter.

Thus multiplying the two Quantities a and c, by the same Quantity d, you will have these two Quantities ad, cd, which are Equimultiples of the two former a, c, which are similar Parts of the Quantities ad, cd, whether d represent an Integer or Fraction.

IV.

Ratio is the Relation of two Quantities of the same kind, compar'd together in regard of their Quantity, to know how and how often one contains or is contained in the other.

Quantities of the same kind are called *Homogeneous*, as two Lines, two Surfaces, two Solids: Quantities of different kinds are called *Heterogeneous*, as a Line and a Surface, and a Solid, *&c.*

The two Homogeneous Quantities compar'd together in a Ratio, are call'd the *Terms of that Ratio*, that that is compar'd is call'd the *Antecedent*, that to which the former is compar'd is call'd the *Consequent*.

Thus in the Ratio of 2 to 3, the Antecedent is 2, the Consequent is 3. This Ratio may easily be comprehended, expressing it Fraction wise, thus, $\frac{2}{3}$, whose Numerator 2 is the Antecedent, and Denominator 3 is the Consequent.

'Tis evident the Terms of a Ratio ought to be Homogeneous, and of a finite Quantity, because otherwise it could not be said how or how often one Quantity is contain'd in another. Which made *Euclid* say, two Quantities have a Ratio, when by Multiplication one may become greater than the other. Then you may see there is no Ratio between a Line and a Surface, because a Line multiplied, that is produced as much as you please, will not have any Breadth, consequently can never equal a Surface, that besides Length has Breadth.

Nor

Nor is there any Ratio between a finite and an infinite Line, tho' theſe two Quantities are Homogeneous, becauſe 'tis a peculiar Property of finite Quantity to meaſure or be meaſured by another, ſo that one may ſay, one is contain'd in the other a certain number of times.

'Tis evident alſo, that to find the Ratio of one Quantity to another, you muſt divide the Antecedent by the Conſequent, and the Quotient, call'd the *Quantity of the Ratio*, ſhows the Relation of the Antecedent to the Conſequent, or the relative Quantity of the Antecedent in regard of the Conſequent, which is properly call'd *Ratio*.

Since therefore a Ratio is a Quantity or Magnitude, tho' relative, all that agrees to Quantity or Magnitude in general, agrees alſo to a Ratio: Hence a Ratio is divided into a *Ratio of Equality*, and a *Ratio of Inequality*, and one Ratio may be equal or greater than another. But you muſt take care you don't confound the *Ratio of Equality*, with the Equality of two Ratio's; becauſe,

A Ratio of Equality is a Ratio wherein the Antecedent is equal to the Conſequent, as the Ratio of 4 to 4, of B to B, *&c.*

A Ratio of Inequality is a Ratio wherein the Antecedent is greater or leſs than the Conſequent, which from hence is divided into a *Ratio of leſs Inequality, and a Ratio of greater Inequality*.

A Ratio of leſs Inequality is a Ratio wherein the Antecedent is leſs than the Conſequent; as the Ratio of 2 to 3. 'Tis evident from what has been ſaid before, that the Quantity of a ſimilar Ratio, is a Number expreſſing how and how often the Antecedent is contained in the Conſequent, or which is the ſame thing, what Part it is of the Conſequent.

Thus the Ratio of 6 to 12 is an half, becauſe 6 is half 12, and this Ratio is call'd *Subduple*. After the ſame manner the Quantity of the Ratio of 2 to 6 is a third, becauſe 2 is a third of 6, and this Ratio is call'd a *Subtriple*. Thus alſo the Quantity of the Ratio of 4 to 6, is two thirds, becauſe 4 is equal to two thirds of 6, and this Ratio is call'd a *Subſeſquialter*, becauſe 4 is contain'd in 6, once and half a time more.

A Ratio of greater Inequality, is a Ratio wherein the Antecedent is greater than the Conſequent; as the Ratio of 3 to 2. 'Tis evident from what has been ſaid above, that the Quantity of a like Ratio is a Number expreſſing how and how often the Antecedent contains the Conſequent, or which is the ſame thing, what Part of the Antecedent the Conſequent is.

Thus the Quantity of the Ratio of 12 to 6 is 2, because 12 contains 6 twice, and this Ratio is call'd the *Duple*. After the ſame manner, the Quantity of the Ratio of 6 to 2 is 3, becauſe 6 contains 2 three times, and this Ratio is call'd the *Triple*. In like manner the Quantity of the Ratio of 6 to 4 is one and an half, becauſe 6 contains 4 once and an half, and this Ratio is call'd *Seſquialter*.

The Ratio of Inequality is divided further into that which is called *Number to Number*, and that which is call'd a *Surd Ratio*.

The Ratio of Number to Number is call'd *a Rational Ratio*, and is ſuch an one as may be expreſſed in Numbers, that is you may expreſs by Numbers how often the Antecedent contains or is contain'd in the Conſequent. Such is the Ratio of a Foot to a Yard, becauſe a Foot is to a Yard as 1 to 3, or the Antecedent is contain'd 3 times in the Conſequent. Such is alſo the Ratio of a Line of 6 Feet to a Line of 4 Feet, where the Antecedent contains the Conſequent once and an half.

A Surd Ratio, call'd alſo an *Irrational Ratio*, is that which can't be expreſſed in Numbers; that is to ſay, 'tis impoſſible to expreſs by Numbers how often the Antecedent is contained, or does contain the Conſequent, as the Ratio of the Side of a Square to its Diagonal, which is ſuch, that tho' each Line apart has aliquot Parts, leſs and leſs continually, yet not one of thoſe that meaſures for Inſtance the Side of the Square, tho' taken never ſo ſmall, can meaſure the Diagonal exactly, that is to ſay, that it ſhall be contain'd in it a certain Number of Times without a Remainder, which is the Reaſon why the Ratio of thoſe two Lines can't be expreſſed in Numbers.

When the Ratio of two Quantities is that of Number to Number, the Quantities are ſaid to be *Commenſurable*, becauſe they have ſome kind of Part that may ſerve as a common meaſure; but if the Ratio of two Quantities be irrational, becauſe they have no Part ſo ſmall as to be a common meaſure to both Quantities, then they are call'd *Incommenſurable*.

The Ratio we have already ſpoken of at preſent, and ſhall further treat of, is call'd *Geometric Ratio*, to diſtinguiſh it from *Arithmetick Ratio*, which is the Relation of two Homogeneous Quantities, conſidering how much one exceeds or is exceeded by another, when they are unequal, which is call'd their *Difference*. When *Ratio* is

mention'd

mention'd alone, you muſt underſtand *Geometric*, concerning which *Euclid* deſigns to ſpeak in theſe Elements.

V.

Equal or *ſimilar* Ratio's are ſuch as have their Antecedents equally containing or contained in their Conſequents, or which is the ſame thing, the Antecedent of one Ratio contains any kind of aliquot Part of its Conſequent, as often as the Antecedent of the other Ratio contains a ſimilar aliquot part of its Conſequent.

Thus the Ratio of 2 *to* 3, *is the ſame or equal or ſimilar to the Ratio of* 4 *to* 6, *becauſe* 2 *is in* 3 *once and an half, and in like manner* 4 *is in* 6, *once and an half; or* 2 *contains two thirds of* 3, *as well as* 4 *contains* 2 *thirds of* 6.

This is the Reaſon why we ſay 2 is to 3, as 4 is to 6, and for brevity uſe four Points :: to expreſs the Equality of the two Ratio's, writing it thus, 2, 3 :: 4, 6, to ſignify that the Ratio of 2 to 3, is equal to the Ratio of 4 to 6. In like manner to expreſs that *a* is to *ad*, as *b* is to *bd*, we write thus *a*, *ad* :: *b*, *bd*.

VI.

Proportional Quantities are ſuch as have the ſame Ratio; *ſuch are the four following*, 2, 3, 4, 6, *becauſe the Ratio of* 2 *to* 3 *is the ſame as that of* 4 *to* 6; *as alſo the four following* a, ad, b, bd, *becauſe the firſt* a *is contain'd as often in the ſecond* ad. *as the third* b *is in the fourth* bd, *the equal Number of Times being repreſented by the ſame Letter* d, *which may be taken for an Integer or Fraction.*

VII.

That Ratio is greater than another, whoſe Antecedent contains any aliquot Part of its Conſequent, oftner than the Antecedent of the other contains a ſimilar aliquot Part of its Conſequent. *Thus the Ratio of* 101 *to* 10 *is greater than the Ratio of* 500 *to* 50, *becauſe* 101 *contains a hundred and one times the tenth Part of* 10, *whereas* 500 *contains but one hundred Times the tenth Part of* 50, *that is* 5.

VIII.

Proportion or *Analogy*, which is frequently confounded with Ratio, is a Similitude or Equality of two Ratio's; *for instance* 2, 3, :: 4, 6, *where you see the four proportional Quantities make a Proportion.*

In a Proportion there are always four Terms, the first and fourth, that is, the first Antecedent and the last Consequent, are called Extreams; the second and third, that is, the Consequent of the first Ratio, and Antecedent of the second, are call'd the *Means*; the two Antecedents are called *Homologous Terms*, and so are the two Consequents.

These four Terms may sometimes be reduced to three, as when the Consequent of the first Ratio is the same as the Antecedent of the second, and then the Proportion is call'd *continued*, thus 2, 4 :: 4, 8. But if the four Terms are different, as these are 2, 3 :: 4, 6. 'tis call'd *discontinued Proportion*.

The Proportion that we have and shall treat of here, is call'd *Geometric Proportion*, to distinguish it from *Arithmetic Proportion*, which is an Equality of two Arithmetic Ratio's found between four Quantities, where the first exceeds the second, or is exceeded by it, by a Quantity equal to that, whereby the third exceeds, or is exceeded by the fourth; and sometimes these four Terms also may be reduced to three; but this kind of Proportion not being used in these Elements, I shall only speak of the Geometric, and that under the single Name of *Proportion*.

IX.

Quantities continually proportional, are such as are in a continued Proportion, as 2, 4, 8, or 1, 3, 9, 27, or *aaa*, *aab*, *abb*, *bbb*, &c.

A Series of Quantities continually Proportional, is call'd a *Progression*, and may be either *Geometric*, or *Arithmetic*, as the Quantities are in a continued Geometric or Arithmetic Proportion. Thus the Quantities, 1, 2, 4, 8, 16, 32, *&c.* are a *Geometric Progression*, and the Quantities 1, 3, 5, 7, 9, 11, *&c.* are an *Arithmetic Progression*.

X. In

X.

In a Geometric Progression, that is to say, in a Series of Quantities continually proportional, the Ratio of the first to the third, is the *Duplicate*, the Ratio of the first to the second, or the Ratio of the second to the third, because those two Ratio's are equal; and the Ratio of the first to the fourth is the *Triplicate* of the Ratio of the first to the second, or of the second to the third, or of the third to the fourth, and so on.

Thus in this Series of Quantities continually proportional, 32, 16, 8, 4, 2, 1, *the Ratio of* 32 *to* 8, *is the Duplicate of the Ratio of* 32 *to* 16, *or of the Ratio of* 16 *to* 8. *because it contains these two equal Ratio's; and the Ratio of* 32 *to* 4, *is the Triplicate of the Ratio of* 32 *to* 16, *or of the Ratio of* 16 *to* 8, *or of the Ratio of* 8 *to* 4, *because it contains those three equal Ratio's.*

You must take care not to confound a Duple Ratio, with a Duplicate Ratio, or a Triple Ratio with a Triplicate Ratio. Thus in the foregoing Example, I took notice that the Ratio of 32 to 8, which is Quadruple, is the Duplicate of 32 to 16, which is Duple; and that the Ratio of 32 to 4, which is Octuple, is the Triplicate of the Ratio of 32 to 16, which is Duple, this Triplicate Ratio being so call'd, because 'tis made up of three equal Ratio's, as the first was call'd the Duplicate, because it is made up of two equal Ratio's. This will be better understood, when I have explain'd what a Ratio made up of several others, is.

A Ratio is then said *to be compounded of other Ratio's*, when its Antecedent is equal to the Product of all the Antecedents of the other Ratio's drawn into one another; and its Consequent in like manner, equal to the Product of all the Consequents of the other Ratio's.

Thus the Ratio $\frac{48}{105}$ is compounded, or made up of these three Ratio's $\frac{2}{3}$, $\frac{4}{5}$, $\frac{6}{7}$, that is to say, the Ratio of 48 to 105, or of 16 to 35, taking the third Part of each Term, is compounded of the Ratio of 2 to 3, of the Ratio of 4 to 5, and of the Ratio of 6 to 7, because the Antecedent 48 is equal to the Product of the three Antecedents 2, 4, 6, and the Consequent 105, is equal to the Product of the Consequents 3, 5, 7.

The Necessity of this Multiplication will be evident to any one that considers that a Ratio made up of a Duple and Triple is a Sextuple, whose Quantity 6 is equal to the Product of 2 and 3, the Quantities of the Duple and Triple Ratio's; it being certain that the double of a Triple or the triple of a Duple is a Sextuple, because 2 multiplied by 3, or 3 by 2, makes 6. Whence it follows, that the *Quantity of a Duplicate Ratio*, is a square Number, namely, the Square of the Quantity common to the two equal Ratio's, that make up the Duplicate Ratio; and that the *Quantity of a Triplicate Ratio*, is a Cube, namely the Cube of the Quantity common to the three equal Ratio's, of which the Triplicate Ratio is compounded, and consequently *the Duplicate Ratio of a Duple Ratio* is a Quadruple, because the Square of 2 is 4, and the *Duplicate Ratio of a Triple Ratio* is a Noncuple, because the Square of 3 is 9; and so the *Triplicate Ratio of a Duple Ratio* is Octuple, because the Cube of 2 is 8. And so of the rest.

'Tis easy to see, by what has been said, that the same Ratio may be compounded of several different Ratio's, because several different Quantities, multiplied together may produce the same Number, for the Quantity of the Ratio, that is compounded of them. Thus the Dodecuple Ratio, whose Quantity is 12, is compounded of the Triple and Quadruple, because their Quantities 3 and 4 multiplied together make 12; also of the Duple and Sextuple, because their Quantities 2 and 6 multiplied together, produce the same Number 12. Whence it follows that Ratio's compounded of equal Ratio's are equal.

'Tis evident that in a Series of as many Quantities as you will, the Ratio of the first to the last is compounded of all the particular Ratio's of the first to the second, of the second to the third, of the third to the fourth, and so on to the last, because the Quantities of all these Ratio's multiplied together, produce the Quantity of the Ratio of the first to the last. Thus in these four Quantities a, b, c, d, the Ratio of the first to the last, namely $\frac{a}{d}$ is compounded of $\frac{a}{b}$ the Ratio of the first to the second, of $\frac{b}{c}$ the Ratio of the second to the third, of $\frac{c}{d}$ the Ratio of the third to the fourth, because these three Ratio's, $\frac{a}{b}$, $\frac{b}{c}$, $\frac{c}{d}$, multiplided together make $\frac{abc}{bcd}$, or $\frac{a}{d}$, namely the Ratio of the first to the last.

These

These Remarks serve to demonstrate *Prop.* 22 and 23.

This Book being compos'd principally to demonstrate the remaining Definitions, that serve to argue by Proportion; I thought it better to omit them here, and explain and demonstrate them in their proper Place, in the following Propositions.

PROPOSITION. VII.

THEOREM VII.

Equal Quantities have a like Ratio to the same third Quantity, and the same Quantity has a like Ratio to equal Quantities.

A. 24. B. 24. C. 8. I Say first, that if the two Quantities A and B are equal; they will have the same Ratio to a third Quantity C.

DEMONSTRATION.

Because the two Quantities A, B, are equal by *Sup.* they will contain any aliquot Part of the third Quantity C, the one as often as the other, and so by *Def.* 5. they will have the same Ratio to that third Quantity. *Which was to be demonstrated.*

I say, secondly, that if the Quantities A and B are equal, the Quantity C will have the same Ratio to the Quantity A, as it has to the Quantity B.

DE-

DEMONSTRATION.

Becauſe the two Quantities A, B, are equal, by *Sup.* their ſimilar aliquot Parts will alſo be equal, and the third Quantity C, will contain each of them equally; wherefore by *Def.* 5. that third Quantity C will have the ſame Ratio to each of the two equal Quantities A, B. *Which was to be demonſtrated.*

USE.

This Propoſition ſerves to demonſtrate the 14 *Prop.* of this Book, and 14 and 15 *Prop. Book* 6. and *Prop.* 34. 12.

PROPOSITION VIII.

THEOREM VIII.

Of two Quantities, the greater has a greater Ratio to a third than the leſs: and this third Quantity has a greater Ratio to the leſs, than it has to the greater.

A. 48. B. 36. C. 12. I Say firſt, that if of two Quantities A, B, the greater is A, it will have a greater Ratio to a third Quantity C, than the leſs one B, has.

DEMONSTRATION.

Becauſe the Quantity A is greater than the Quantity B, by *Sup.* it will contain a certain aliquot Part of C, oftner than the Quantity B does, and by *Def.* 7. the Ratio of A to C, will be greater than the Ratio of B to C. *Which was to be demonſtrated.*

I ſay in the ſecond Place, if the Quantity B is leſs than the Quantity A, the Ratio of C to B, is greater than the Ratio of C to A.

DE-

DEMONSTRATION.

Because the Quantity B is less than the Quantity A, by *Sup.* its aliquot Parts will be less than the similar aliquot Parts of the Quantity A, consequently the Quantity C will contain an aliquot Part of the Quantity B, oftner than it will a similar aliquot Part of the Quantity A; wherefore by *Def.* 7. the Ratio of C to B, will be greater than the Ratio of C to A. *Which was to be demonstrated.*

USE.

This Proposition serves to demonstrate *Prop.* 14.

PROPOSITION IX.

THEOREM IX.

Quantities having the same Ratio to a third are equal; and they to which a third Quantity has the same Ratio are also equal.

A. 3. C. 2. B. 3.

I Say first, if each of the two Quantities A, B, have the same Ratio to a third Quantity C, these two Quanties A, B, are equal.

DEMONSTRATION.

Because the Ratio of A to C is equal to that of B to C, by *Sup.* the Quantity A will contain an aliquot Part of the Quantity C, as often as B does, by *Def.* 5. and consequently these two Quantities A and B will be equal. *Which was to be demonstrated.*

I say in the second Place, that if a third Quantity C have the same Ratio to each of the two Quantities A and B, these two Quantities A and B are also equal.

DEMONSTRATION.

Because the Ratio of C to A is equal to that of C to B, by *Sup.* a certain aliquot Part of A will be contain'd in C, as often as a similar aliquot Part of B, by *Def.* 5. Wherefore an aliquot Part of A will be equal to a similar aliquot Part of B, and consequently A and B will be equal. *Which remain'd to be demonstrated.*

USE.

This Proposition serves to demonstrate *Prop.* 14. and *Prop.* 2, 5, 7, 14, 25, and 31. *Book* 6. and *Prop.* 34. *Book* 11. Lastly, *Prop.* 15. *Book.* 12.

PROPOSITION X.

THEOREM X.

Of two Quantities, that which has the greatest Ratio to a third Quantity, is the greater; on the contrary, that to which a third has a greater Ratio, is the less.

A. 12.
B. 8. C. 2.

I Say first, that if of two Quantities A, B, the first A has a greater Ratio to a third Quantity C, than the second B to the same Quantity C, that first Quantity A is greater than the second B.

DEMONSTRATION.

Because the Ratio of A to C is greater than that of B to C, by *Sup.* the Antecedent A contains a certain aliquot Part of its Consequent C, oftner than the Antecedent B contains a similar aliquot Part of its Consequent C, by *Def.* 7. Whence it follows that the Quantity A is greater than the Quantity B. *Which was to be demonstrated.*

I Say, in the second Place, that if the third Quantity C, has a greater Ratio to the second B, than it has to the first A, that second Quantity B, is less than the former A.

DE-

DEMONSTRATION.

Because the Ratio of C to B is greater than the Ratio of C to A, by *Sup.* the Quantity C contains a certain aliquot Part of B, oftner than it does a similar aliquot Part of A, by *Def.* 7. and consequently B will be less than A. *Which remains to be demonstrated.*

USE.

This Proposition serves to demonstrate *Prop.* 14.

PROPOSITION XI.

THEOREM XI.

Ratio's equal to the same Ratio, are equal to one another.

A. 2. B. 3. :: C. 4. D. 6.
E. 8. F. 12. :: C. 4. D. 6.

I Say, if the two Ratio's of A to B, and of E to F, are each equal to that of C to D, they are equal to one another.

DEMONSTRATION.

Because A is to be, as C to D, the Antecedent A contains its Consequent B, as often as the Antecedent C does its Consequent D: likewise because E is to F, as C to D, the Antecedent E will contain its Consequent F, as often as the Antecedent C, does its Consequent D, by *Def.* 5. Wherefore the Antecedent A will contain its Consequent B, as often as the Antecedent E does its Consequent F, and by *Def.* 5. the Ratio of A to B, will be equal to that of E to F. *Which was to be demonstrated.*

USE.

This Proposition serves to demonstrate *Prop.* 25, and 31. *Book* 6. and *Prop.* 34. *B.* 12.

PRO-

PROPOSITION XII.

THEOREM XII.

If several Quantities are proportional, the Sum of all the Antecedents is to the Sum of all the Consequents, as any one Antecedent is to its Consequent.

A. 2. B. 4. :: C. 3. D. 6. I Say, if the Ratio of A to B, be the same as the Ratio of C to D, the Ratio of the Sum A+C of the two two Antecedents, to the Sum B+D of the two Consequents, is the same as that of the Antecedent A to the Consequent B.

DEMONSTRATION.

Because A is to B, as C to D, by *Sup.* the Antecedent A will contain any aliquot Part of its Consequent B, as often as the Antecedent C contains a similar aliquot Part of its Consequent D; for instance an half, by *Def.* 5. and since half B added to half D, makes half B+D, A+C will contain half B+D as often as A contains half B, and consequently the Ratio of A to B, is similar to that of A+C, to B+D. *Which was to be demonstrated.*

USE.

This Proposition serves to demonstrate *Prop.* 5, 6, and 7, *of Book* 12, and that an Ellipse is a mean Proportional between two Circles describ'd about its two Axes, as you will find in our *Planimetry*. It serves also to demonstrate the Rule of Fellowship, and *Prop.* 20. 6. and *Prop.* 25. 12.

PRO-

PROPOSITION XIII.

THEOREM XIII.

If two Ratio's be equal, and one greater than a third Ratio, the other will also be greater than the same third Ratio.

A. 2. B. 3. :: C. 4. D. 6. E. 7. F. 12. I Say, if the two Ratio's of A to B, and of C to D, be equal, and the first Ratio of A to B greater than the third Ratio of E to F, the second Ratio of C to D, will also be greater than the same Ratio of E to F.

DEMONSTRATION.

Because the Ratio of A to B is greater than that of E to F, by *Sup.* the Antecedent A will contain any aliquot Part of its consequent B, oftner than the Antecedent E contains a similar aliquot Part of its Consequent F, by *Def.* 7. and since the Antecedent C contains a similar aliquot Part of its Consequent D, as often as the Antecedent A contains that of its Consequent B, because the Ratio of A to B is the same with that of C to D, by *Sup.* the Antecedent C must contain an aliquot Part of its Consequent D, oftner than the Antecedent E, contains a similar aliquot Part of its Consequent F, and by *Def.* 7. the Ratio of C to D, being also greater than that of E to F. *Which was to be demonstrated.*

PROPOSITION XIV.

THEOREM XIV.

In four proportional Quantities, if the first be greater, equal, or less than the third, the second also will be greater, equal, or less than the fourth.

A, B. :: C, D. 12. 3. :: 4. 1. I Say first, if of these four Proportional Quantities, A, B, C, D, the first, which is A, be greater than the third, C, the second also, B, will be greater than the fourth D.

DE-

DEMONSTRATION.

Becauſe A is greater than C, by *Sup.* the Ratio of A to B is greater than the Ratio of C to B, by *Prop.* 2. and ſince the Ratio of A to B is equal to that of C to D, by *Sup.* the Ratio of C to D will be greater than that of C to B, and by *Prop.* 4. B will be greater than D. *Which was to be demonſtrated.*

A. B. :: C. D. I ſay, ſecondly, if A the firſt of theſe
3. 4. :: 3. 4. four proportional Quantities, A, B, C, D, be equal to C the third, B alſo the ſecond will be equal to D the fourth.

DEMONSTRATION.

Becauſe A is equal to C by *Sup.* the Ratio of A to B, is the ſame as that of C to B, by *Prop.* 1. and ſince the Ratio of A to B, is equal to that of C to D by *Sup.* the Ratio of C to D will be the ſame as that of C to B, and by *Prop.* 3. B will be equal to D. *Which was to be demonſtrated.*

A. B. :: C. D. Laſtly, I ſay if A, the firſt of theſe
3. 4. :: 3. 6. four proportional Quantities, A, B, C, D, be leſs than C the third, B the ſecond will be alſo leſs than D the fourth.

DEMONSTRATION.

Becauſe A is leſs than C by *Sup.* the Ratio of A to C will be leſs than that of C to B, by *Prop.* 2. and ſince the Ratio of A to B is equal to that of C to D, by *Sup.* the Ratio of C to D will be leſs than that of C to B, and by *Prop.* 4. B will be leſs than D. *Which remain'd to be demonſtrated.*

USE.

It ſerves to demonſtrate *Prop.* 24. and *Prop.* 25. 15 and 25, *of Book* 6.

LEM-

LEMMA I.

If four Quantities be proportional, the Product of the Extreams is equal to the Product of the Means.

THese four Quantities a, ad, b, bd, *being proportional, by* Def. 6. *the Product of the two Extreams* a, bd, *is evidently equal to the Product of the Means,* ad, b, *because the two Extreams* a, bd, *multiplied together are equal to the two Means* ad, b, *multiplied together; namely,* abd. Which was to be demonstrated.

LEMMA II.

Those four Quantities are proportional, the Product of whose Extreams is equal to the Product of the two Means.

I *Say, these four Quantities* a, b, c, d, *are proportional, if the Product* ad *of the Extreams be equal to* be *the Product of the Means.*

DEMONSRATION.

Suppose a *to be contain'd in* b, *a certain Number of Times expressed by* m, *in which Case* am *will be equal to* b, *and* c *contain'd in* d, *a certain Number of Times expressed by* n, *then* cn *will be equal to* d, *instead of having the Product* ad, *equal to the Product* bc, *you will have the Product* acn *equal to the Product* acm; *consequently dividing each of the equal Terms by* ac, *you will have* m *equal to* n; *wherefore* b *contains* a *as often as* d *does* c, *and by* Def. 6. *the four Quantities* a, b, c, d, *are proportional.* Which was to be demonstrated.

PROPOSITION XV.

THEOREM XV.

Equimultiples, and their similar Aliquot Parts, are proportional.

I Say, the four Quantities *ad, bd, a, b,* whose two first Terms *ad, bd,* are Equimultiples of the two last, *a b,* are proportional.

DEMONSTRATION.

Because the Product of the two Extreams *ad*, *b*, of the four Quantities proposed *ad*, *bd*, *a*, *b*, is the same with the Product of the two Means *bd*, *a*, namely *abd*, consequently by *Lemma* 2. the four Quantities *ad*, *bd*, *a*, *b*, are proportional. *Which was to be demonstrated.*

USE.

This Proposition serves to demonstrate *Prop.* 1. and 33. *Book* 6. and *Prop.* 13. 12.

PROPOSITION XVI.

THEOREM XVI.

If four Quantities are proportional, they are also proportional when altern'd.

A Ratio is said to be altern'd, when the Place of the two middle terms in the Proportion is chang'd, the one being substituted in the room of the other, and the Proportion yet continuing; that is to say, the four Quantities that were proportional, continue to be so after this Change: But this is to be demonstrated.

A, B. : : C, D.
2. 3. : : 4. 6.

I say therefore, if the four Quantities A, B, C, D, are proportional, these four A, C, B, D, are proportional also.

DEMONSTRATION.

For since the four Quantities A, B, C, D, are proportional, by *Sup.* by *Lem* 1. the Product AD of the Extreams, is equal to the Product BC of the Means; and by *Lem.* 2. these four Quantities A, C, B, D, are also proportional. *Which was to be demonstrated.*

Or

Or because the Ratio of A to C is compounded of the Ratio's of A to B, and of B to C, which are equal to the two Ratio's of B to C, and of C to D, of which the Ratio also of B to D is compounded: 'Tis easy to conclude from the Remarks made in *Def.* 10. that the Ratio of A to C is equal to that of B to D, that is to say, that the four Quantities A, C, B, D, are proportional. *Which was to be demonstrated.*

SCHOLIUM.

An inverted Ratio.

One may demonstrate after the same manner, what *Euclid* demonstrates after the 4th *Prop.* which we have omitted, namely, that if the four Quantities A, B, C, D, are proportional, these four also B, A, D, C, are also proportional, which is call'd an *inverted Ratio*, in which we compare the Consequent with the Antecedent; because the Quantities A, B, C, D, being proportional, the Product AD of the two Extreams is equal to the Product of the Means BC, by *Lem.* 1. and by *Lem.* 2. these four Quantities B, A, D, C, are proportional also.

PROPOSITION XVII.

THEOREM XVII.

Proportion by Division.

If four Quantities are proportional, they will be so also when divided.

A *Proportion* is said to be *divided*, when instead of each Antecedent you substitute the Excess of that Antecedent above its Consequent, and still the Quantities are proportional, as we are now to demonstrate.

I say then, if these four Quantities *ad*, *a*, *bd*, *b*, are proportional, as they certainly are, as 'tis evident by *Def.* 6. and also by *Lem.* 2. that is to say, the Ratio of *ad* to *a* is the same as that of *bd* to *b*; by dividing the Proportion, the Ratio of *ad—a* to *a*, is the same with that of *bd—b* to *b*.

DEMONSTRATION.

Because the Product of the two Means *a*, *bd*—*b*, and of the two Extreams *ad*—*a*, *b*, of these four Quantities, *ad*—*a*, *a*, *bd*—*b*, *b*, is the same, namely, *abd*—*ab*, it follows by *Lem.* 2. that these four Quantities *ad*—*a*, *a*, *bd*—*b*, *b*, are proportional. *Which was to be demonstrated.*

SCHOLIUM.

Conversion of Proportion.

The Division of Proportion just now defined, supposes the Antecedent is greater than its Consequent; but since it may be less, and then Proportion by Division seeming impossible, it must be defined more generally, taking the Difference between the Antecedent and Consequent, instead of the Excess, and then if you compare it with the Antecedent, which is call'd *Converting a Proportion*, you may demonstrate that the Proportion remains.

PROPOSITION XVIII.

THEOREM XVIII.

Composition of Proportion.

If four Quantities are Proportional, they are so when Compounded.

When *a Proportion* is said to be *Compounded*, when the Sum of the Antecedent and its Consequent is substituted in the room of each Antecedent, the Quantities continuing to be proportional, as we shall demonstrate.

I say then, if these four Quantities *a*, *ad*, *b*, *bd*, are proportional, as they certainly are, as is evident by *Def.* 6. and *Lem.* 2. that is to say, the Ratio of *a* to *ad*, is the same as that of *b* to *bd*, compounding them the Ratio of *a*+*ad* to *ad*, is the same as that of *d*+*bd*, to *bd*.

DEMONSTRATION.

Because if you multiply the two Extreams *a*+*ad*, *bd* together, and the two Means *ad*, *b*+*bd* of these four proportional

proportional Quantities $a+ad$, ad, $b+bd$, bd, the Product will be the same, namely, $abd+abdd$, consequently by *Lem.* 2. these four Quantities $a+ad$, ad, $b+bd$, bd, are proportional. *Which was to be demonstrated.*

SCHOLIUM.

One might also put instead of each Consequent, the Sum of the Consequent and its Antecedent, to compare it with its Antecedent, and demonstrate after the same manner that the Proportion continues: which *Euclid* demonstrates by a Consequence drawn from *Prop.* 19. which being thus useless, as well as *Prop.* 20. and 21. we shall consequently omit them.

USE.

This Proposition serves to demonstrate *Prop.* 24. and *Prop.* 31. 6.

PROPOSITION XXII.

THEOREM XXII.

Proportion ex æquo ordinata.

If there be a certain Number of Quantities in one Rank in Proportion ex æquo, *with a like Number of Quantities in another, the Ratio of the two Extreams of one Rank is equal to the Ratio of the two Extreams of the other.*

Quantities are said to be in *Proportion ex æquo*, when in several Quantities in one Rank proportional to as many in the other, the first Quantity in one Rank is to the second, as the first in the other Rank is to its second, and the second of the first Rank is to its third, as the second of the second Rank is to its third, and so on.

A. 2. B. 3. C. 4.
D. 8. E. 12. F. 16.

Thus if you have these three Quantities A, B, C, in one Rank, and three others D, E, F, in another, so that A be to B, as D to E, and B to C as E to F, I say then that A is to C as D to F.

DEMONSTRATION.

Becauſe the Ratio of A to C is compounded of the Ratio's of A to B, and of B to C, and the Ratio of D to F, is compounded of the Ratio's of D to E and E to F, which are by *Sup.* equal to the two Ratio's of A to B, and of B to C, it follows that the two Ratio's of A to C, and D to F is compounded of ſimilar Ratio's, and conſequently equal. *Which was to be demonſtrated.*

USE.

This Propoſition ſerves to demonſtrate *Prop. 8. 6.* and ſeveral other fine Theorems in Geometry, as the 4th *Lem. of our Dialling.*

PROPOSITION XXIII.

THEOREM XXIII.

Proportion ex æquo perturbata.

If there be a certain Number of Quantities in one Rank, in a Proportion ex æquo perturbata, *with an equal Number of Terms in another Rank, the Ratio of the two Extreams of one Rank, is equal to the Ratio of the two Extreams of the other Rank.*

A *Proportion* is ſaid to be *ex equo perturbata,* when ſeveral Quantities in one Rank, are proportional to as many in another Rank, ſo as that the firſt of one Rank is to the ſecond, as the laſt ſave one of the other Rank is to the laſt, and the ſecond of the firſt Rank is to the third, as the laſt ſave two of the ſecond Rank is to the laſt ſave one, and ſo on to the firſt of the ſecond Rank.

A. 2. B. 4. C. 1.
D. 12. E. 3. F. 6.

Thus if you have the three Quantities A, B, C, in one Rank, and three others D, E, F, in another, ſo as that A is to B, as E is to F, and B is to C, as D is to E. I ſay in this caſe A is to C, as D is to F.

DEMONSTRATION.

Because the Ratio of A to C is compounded of the Ratio's of A to B and of B to C, and the Ratio of D to F is compounded of the Ratio of D to E, equal to that of B to C, by *Sup.* and of E to F, equal to that of A to B, it follows from the Remarks made on *Def.* 10. that the Ratio of A to C is equal to that of D to F. *Which was to be demonstrated.*

USE.

This Proposition is used in *Spherical Trigonometry,* to demonstrate that *in a Spherical Triangle, the Sines of the Angles are proportional to the Sines of their opposite Sides.* It serves also in *Plain Trigonometry* to demonstrate that in a *Rectilineal Triangle, the Sines of the Angles are proportional to their opposite Sides.* This Proposition is of use also in the Demonstration of *Prop.* 24.

PROPOSITION XXIV.

THEOREM XXIV.

If of six Quantities, the first is to the second as the third is to the fourth; and the fifth to the second, as the sixth to the fourth; the Sum of the first and fifth will be to the second, as the Sum of the third and sixth to the fourth.

A. 2. B. 3. :: C. 4. D. 6.
E. 8. B. 3. :: F. 16. D. 6.

I Say, if of these six Quantities A, B, C, D, E, F, the Ratio of the first A, and second B, be equal to the Ratio of the third C, and fourth D; and the Ratio of the fifth E, to the second B, is the same with that of the sixth F, to the fourth D; the Sum A+E of the first and fifth is to the second B, as the Sum of the third and sixth C+F to the fourth D.

DEMONSTRATION.

Since by *Sup.* the Ratio of A to B, is equal to that of C to D, the Antecedent A, will contain an aliquot Part of its Consequent B, as often as the Antecedent C contains a similar aliquot Part of its Consequent B, by *Def.* 5, and by the same Definition, since the Ratio of E to B is like that of F to D by *Sup.* the Antecedent E will contain the same aliquot Part of its Consequent B, as often as the Antecedent F contains a similar aliquot Part of its Consequent D: Consequently A+E, the Sum of the two Antecedents A, E, will contain any aliquot Part whatever of their common Consequent B, as often as C+F, the Sum of the two other Consequents C, F, contains a similar aliquot Part of their common Consequent D: and so by *Def.* 5. the Ratio of A+E to B, will be the same as that of C+F to D. *Which was to be demonstrated.*

SCHOLIUM.

This Proposition may be demonstrated otherwise and easier thus: Since the Ratio of E to B, is supposed equal to that of F to D, *by Inversion of Proportion*; the Ratio of B to E is the same with that of D to F; and since the Ratio of A to B, is the same with that of C to D, by Supposition, you will have these three Quantities A, B, E, in one Rank, and C, D, F, in another, in a Proportion *ex æquo ordinata*, consequently by *Prop.* 22. the Ratio of A to E is the same with that of C to F, and by Composition of Proportion according to *Prop.* 18. the Ratio of A+E, to E, is the same with that of C+F, to F. *Which was to be demonstrated.*

PRO-

PROPOSITION XXV.

THEOREM XXV.

In four proportional Quantities the Sum of the two Extreams is greater than the Sum of the two Means.

I Say, the Sum of the two Extreams *ab*+*cd*, of these four Quantities *ab*, *bd*, *ac*, *cd*, proportional by *Lem.* 2. is greater than *ac*+*bd*, the Sum of the two Means.

DEMONSTRATION.

If the first *ab* be supposed greater than the third *ac*, divide each of those two unequal Quantities *ab*, *ac*, by *a*, and you will find the Quantity *b* is greater than the Quantity *c*, then multiply each of these two unequal Quantities, *b*, *c*, by the Difference *a*—*d*, and you will find the Product *ab*−*bd*, greater than the Product *ac*−*cd*; and lastly, add to each of these unequal Products, *ab*−*bd*, *ac*—*cd*, the Sum *bd*+*cd*, you will find the Sum *ab*+*cd*, is greater than the Sum *ac*+*bd*. *Which was to be demonstrated.*

SCHOLIUM.

If you would have another Demonstration, suppose the four Quantities, A, B, C, D, proportional, and the first A greater than the third C, and then the second C, will be greater than the fourth D, by *Prop.* 14. Then, I say, the Sum A+D of the two Extreams is greater than the Sum of the two Means B+C.

DEMONSTRATION.

Since the four Quantities A, B, C, D, are supposed proportional, by Division of Proportion, according to *Prop.* 17. A—B, B, C—D, D, are also proportional; and since we know that B the second, is greater than D the fourth, then by *Prop.* 14. A—B, the first, must be greater than C—D the third; consequently add B+D the Sum to each of these unequal Quantities A—B, C—D, and you will find the Sum A+D, is greater than the Sum B+C. *Which was to be demonstrated.*

USE

USE.

This Proposition serves to shew the Difference between Geometric and Arithmetic Proportion, in the latter, the Sum of the two Extreams is equal to the Sum of the Means, as shall be demonstrated in our Trigonometry; whereas in the former the Sum of the two Extreams is greater than the Sum of the two Means, as has been demonstrated two ways.

The Commentators upon Euclid, *have added nine Propositions more, which we shall omit, because they are not* Euclid'*s, and may be easily understood by any one that understands the foregoing.*

The

THE SIXTH BOOK OF *EUCLID's* ELEMENTS.

Euclid, having explain'd in general the several Sorts of Proportion, begins in this Book to apply them to Planes, and first to Triangles, comparing their Areas, Sides, and Angles respectively together. On that Account this Book is the Foundation of the Construction and Use of all Sorts of Mathematical Instruments, as the Graphometer, Astrolabe, Geometrical Quadrant, Jacob's Staff, Sector, and all others as are of use in Mensuration: and besides of all Machines as are used in Mechanics, instead of moving Powers, as the Balance, Lever, Pully, Axis in Peritrochio, the Screw and the rest as well simple as compound, as serve to augment the Motive forces in any Ratio.

DEFINITIONS.

I.

Similar Rectilineal Figures are such as have all their Angles respectively equal, and the Sides contain'd by them proportional.

This

Plate 1. Fig. 1. *Thus the two Rectilineal Figures ABC, BDE are similar, because the Angle ABC is equal to the Angle BDE, and the Angle BAC equal to the Angle DBE; and the Side AB to the Side BC, as the Side BD, to the Side DE: and the Side AB, to the Side AC, as the Side BD to BE,* &c.

If all the Rectilineal Figures were Triangular, it would be enough to say they are equiangular instead of similar, because in *Prop.* 4. we have demonstrated that equiangular Triangles, have also their Sides proportional; or instead of saying Triangles are similar, one might say they have their Sides proportional, because Triangles that have their Sides proportional, are equiangular, as shall be demonstrated in *Prop.* 5.

II.

Reciprocal Figures are such as have Sides that may be so

Fig. 2. compar'd, as that the Antecedent of one Ratio, and Consequent of the other, is to be found in the same Figure.

Thus the two Figures ABE, ACD, are reciprocal, because as the Side AB is to the Side AC, so is the Side AD to the Side AE.

III.

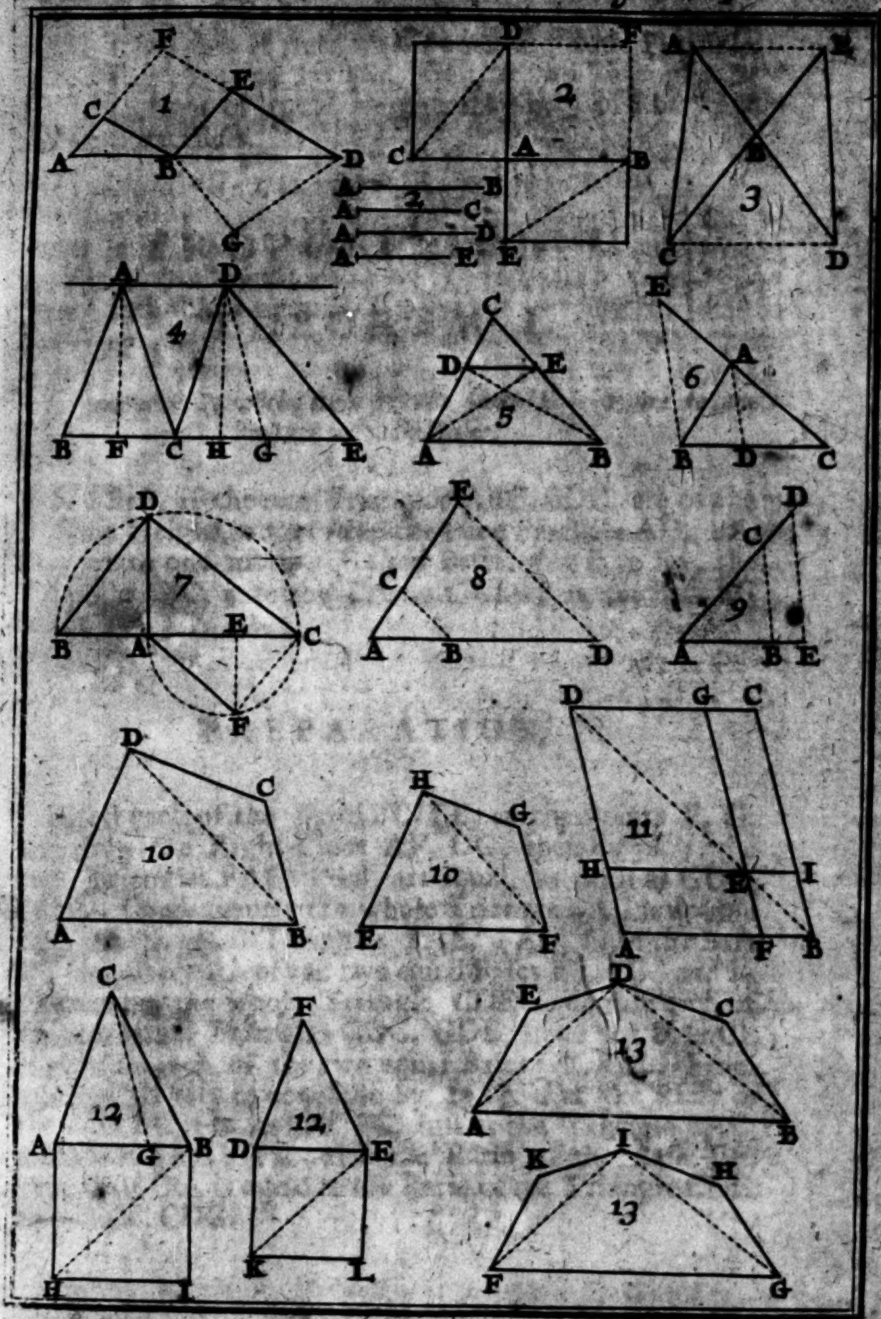
Plate 2. Fig. 18. A Line is said to be cut in *extream and mean Proportion*, when the whole Line is to its greater Part, as that greater Part is to the less. *Thus the Line AD is divided at the Point B, into extream and mean Proportion, if the Ratio of the Line AD, to its greater Part AB, be the same with that of the greater Part AB, to its less BD.*

This Line is so call'd, because in the three Proportionals AD, AB, BD, the *Extream Ratio*, is that between the two Extreams AD, BD, and the *Mean Ratio* is that between the whole AD, and the Mean AB, or between the Mean AB, and the other Extream BD.

IV.

Plate 1. Fig. 4. *The Height of a Figure*, is a Right-Line let fall perpendicularly from the Vertex to the Base. *Thus the Height of the Triangle ABC, is the perpendicular AF, let fall from the Vertex A upon the Base BC: and so also the Height of the Triangle CDE is the perpendicular DH, let fall from the Vertex D upon the Base CE.*

'Tis



'Tis evident, that if two Triangles or Parallelograms of the same Height, have their Bases in the same Right-Line, and the same Way, they are between the same Parallels; and that if they are between the same Parallels, they are of the same Height. So that two Triangles, or Parallelograms of equal Heights may be plac'd between the same Parallels.

PROPOSITION I.

THEOREM I.

Triangles and Parallelograms of the same Height are to one another as their Bases.

I Say first, if the two Triangles ABC, CDE, are of the same Height, or between the same Parallels AD, BE, they are to one another as their Bases, that is to say, the Triangle ABC, is to the Triangle CDE, as the Base BC to the Base CE. Plate 1. Fig. 4.

PREPARATION.

Bisect each of the Bases BC, CE, at the Points F, G, and draw the Right-Lines AF, DG; then by 38. 1. the two Triangles FAC, FAB, are equal, as well as GDC, GDE. Consequently the whole Triangle BAC is double each of the equal Triangles FAB, FAC, since the Base BC is double each of the two equal Bases FB, FC: and in like manner the whole Triangle CDE is double each of the two equal Triangles GDC, GDE, since the Base CE is double each of the two equal Bases GC, GE. From whence 'tis easy to conclude by 15. 5. that the Ratio of the Base BC is to its half FC, just as the Triangle BAC, to its half FAC: Thus also the Ratio of the Base CE, to its half CG, is equal to the Ratio of the Triangle CDE, to its half CDG.

DE-

DEMONSTRATION.

This being supposed, consider BC is to its half FC, as CE to its half CG: and so also that the Triangle BAC, is to its half FAC, as the Triangle CDE, to its half CDG, and consequently the Proportion between the four Lines BC, FC, CE, CG, is similar to the Proportion that is between the four Triangles BAC, FAC, CDE, CDG: Wherefore changing them by 16. 5. You will find the Heights AF and DH being equal, that the Proportion between the four Lines BC, CE, CF, CG, is equal to that between the four Triangles BAC, CDE, FAC, CDG. Whence 'tis easy to conclude that in this second Proportion, the first Triangle BAC is to the second CDE, as the first Line BC, to the second Line CE, in the first Proportion. *Which was to be demonstrated.*

I say in the second Place, that Parallelograms of the same Height are to one another as their Bases, because Parallelograms being double Triangles of the same Base and Height, by 41. 1. are as their Bases, *&c. Which remain'd to be demonstrated.*

USE.

This Proposition is of use in the following, and in *Prop.* 14, 15, and 19. and also to demonstrate that Triangles and Parallelograms, whose Bases are equal, are as their Heights, because their Heights may be taken for their Bases, and the Bases for Heights, which is too easy to insist upon.

PRO-

PROPOSITION II.

Plate I. Fig. 5.

THEOREM II.

A Right-Line drawn Parallel to one of the Sides of a Triangle, cuts the Legs proportionally; and if it cuts the Legs proportionally, 'tis parallel to the third Side.

I Say first, if the Right-Line DE be drawn parallel to the Side AB, of the Triangle ABC, it will cut the two other Legs AC, BC, proportionally, so that the Part CD shall be to the Part AD, as the Part CE to the Part BE.

DEMONSTRATION.

Draw the Right-Lines AE, BD, and you will find the two Triangles CED, DEA, having the same Vertex E, to have the same Height, and by *Prop*. 1. they are to one another as their Bases CD, AD: After the same manner, the two Triangles CDE, EDB, having the same Vertex D, and consequently the same Height, are to one another as their Bases, CE, BE; and since the two Triangles DEA, EDB, between the same Parallels AB, DE, and having the same Base DE, are equal by 37. 'Tis easy to conclude by 11. 5. the Ratio of the Parts CD, AD, is the same with that of the Parts CE, BE, *Which was to be demonstrated.*

I say secondly, if the Line DE, cut the two Sides AC, BC, proportionally, 'tis parallel to the third Side AB.

DEMONSTRATION,

Connecting as before, the Right-Lines AE, BD, consider that since the four Lines CD, AD, CE, BE, are proportional by *Sup*. the four Triangles CED, DEA, CDE, EDB, are proportional by *Prop*. 1. and because the

Plate 8. Fig. 5. the two Antecedents CED, and CDE, are equal, representing the same Triangle, the Consequents also are equal, DEA, EDB, by 14. 5. Wherefore by 39. 1. the Line DE will be parallel to the Side AB. *Which remain'd to be demonstrated.*

USE.

This Proposition serves to demonstrate the following one and *Prop.* 4. and that several Lines drawn Parallel to the same Side cut the Legs proportionally.

PROPOSITION III.

THEOREM III.

A Right-Line bisecting an Angle of a Triangle, divides the opposite Side into two Parts that are in the same Ratio as the two other Sides: and if it divide a Side into two Parts proportional to the two other Sides, it bisects the opposite Angle.

Fig. 6. I Say first, if the Right-Line AD, bisect the Angle BAC of the Triangle, it cuts the opposite Side BC into two Parts BD, CD, that are in the same Ratio as the two other Sides AB, AC.

PREPARATION.

Produce one of the two Sides AB, AC, as AC in E, 'till AE be equal to the other Side AB, and join the Right-Line BE.

DEMONSTRATION.

Because the Triangle BAE is an Isoscele, by *Const.* the Angle E will be equal to the Angle ABE, by 5. 1. and because the external Angle BAC, double the Angle BAD, is equal to the two internal and opposite E,

E, ABE, by 32. 1. it will be double each, and consequently the Angle ABE. So the alternate Angles BAD, ABE, will be equal, and by 27. 1. the Line AD will be parallel to the Side BE of the Triangle BEC, and by *Prop.* 2. the Ratio of the two Parts BD, CD, will be equal to that of the two Parts AE, AC, or the two Sides AB, AC. *Which was to be demonstrated.* Plate 1. Fig. 6.

I say secondly, if the Ratio of the two Parts BD, CD, be equal to that of the two Sides AB, AC, the Angle BAD is equal to the Angle CAD.

DEMONSTRATION.

Make a Construction similar to the foregoing, and since by *Sup.* the Ratio of the two Lines BD, CD, is equal to that of the two AB, AC, or AE, AC, the Line AD is parallel to the Side BE of the Triangle AEB, by *Prop.* 2. and by *Prop.* 29. 1. the Angle BAD is equal to each of the two equal Angles E, ABE; and since the Angle BAC is double the Angle E, it will be also double the Angle BAD, which will consequently be equal to the Angle CAD. *Which remain'd to be demonstrated.*

USE.

This Proposition may serve to divide a given Line into two Parts proportional to two other given Lines; provided the Sum of the two given Lines be greater than the first: Thus to cut the Line BC into two Parts proportional to the two given Lines AB, AC, form with the three given Lines BC, AB, AC, the Triangle BAC, by 22. 1. and by 19. 1. bisect the Angle A, by the Right-Line AD, &c.

PROPOSITION IV.

THEOREM IV.

Equiangular Triangles have their Sides proportional.

I Say, if the two Triangles ABC, BDE, are equiangular, so that the Angle A, is equal to the Angle DBE, and the Angle ABC equal to the Angle BDE, and consequently the third Angle ACB equal to the third Fig. 2.

 Angle

Plate I. Fig. I. Angle BED; the Ratio of the two Sides AB, BD, opposite to the equal Angles, will be equal to that of the two Sides BC, DE, opposite to the equal Angles; and in like manner the Ratio of the two Sides AB, AD, opposite to the equal Angles, is equal to that of the two Sides AC, BE, opposite to equal Angles.

PREPARATION.

Having imagin'd the two Triangles ABC, BDE, so posited that the two Sides opposite to the equal Angles, as AB, BD, join by their Extremities in a Right-Line, produce the two Sides AC, DE, 'till they meet in a Point, as F.

DEMONSTRATION.

Because ABD is a Right-Line, and by *Const.* the Angle ADF, equal to the Angle ABC, by *Sup.* the Line BC will be parallel to the Line DF, by 28. 1. and so also because the Angle A is equal to the Angle DBE, the the Line BE will be parallel to the Line AF: Thus the Figure BCFE will be a Parallelogram, whose two opposite Sides BC, EF, are equal, by 34. 1. as well as the two opposite ones, BE, CF, and in the Triangle ADF, the Line BC being parallel to the Side DF, the Ratio of AB to BD will be equal to that of AC to CF, or BE, by *Prop.* 2. and so also the Line BE being parallel to the Side AF, the Ratio of the two Lines AB, BD is equal to that of those two EF or BC, and DE. *Which was to be demonstrated.*

SCHOLIUM.

'Tis evident by 11. 5. that the Ratio of the two Sides AC, BE, opposite to the equal Angles, is also equal to that of the two Sides BC, DE, opposite to equal Angles, because each of the two Ratio's has been demonstrated to be equal to that of AB to BD.

'Tis evident also by 16. 5. that the Sides containing the equal Angles in each Triangle, are proportional, that

that is to say, for instance, that the Ratio of the two Sides AB, AC, is equal to that of the two BD, BE, because it has been demonstrated that the four Sides AB, BD, AC, BE, are proportional, consequently by conversion, AB, AC, BD, BE, also are proportional. Whence it follows by *Def.* 1. that equiangular Triangles are similar. Plate 2. Fig. 2.

USE.

This Proposition is not only necessary for the following ones, but is the Foundations of the Principal Practices of Trigonometry, and of the use of the Universal Instrument, on which are described little Triangles, similar to those that are imagin'd to be on the Ground, when 'tis used to measure any inaccessible Line, take a Plan, or trace one upon the Ground: 'Tis also the Foundation of the Use of the Compass of Proportion as may be seen in a Treatise upon that Subject already published, where Demonstrations are founded upon that Proposition.

PROPOSITION V.

THEOREM V.

Triangles that have their Sides proportional, are equiangular.

I Say, if in the two Triangles ABC, BDE, the Side AB, is to the Side BC, as the Side BD to the Side DE: and the Side AB, to the Side AC, as the Side BD, to the Side BE; these two Triangles ABC, BDE, are equiangular, so that the Angle ABC is equal to the Angle BDE, the Angle A to the Angle DBE, and consequently the third Angle ACB, equal to the third Angle BED. Fig. 2.

PREPARATION.

Make by 23. 1. at the Extremity B of the Side BD, the Angle DBG, equal to the Angle A, and at the other Extremity D, the Angle BDG equal to the Angle ABC.

DEMONSTRATION.

Plate 1. Fig. 1. Because the Triangles ABC, BGD, are equiangular by *Const.* the Ratio of AB to BC is the same as that of BD to DG, by *Prop.* 4. and because the Ratio of AB to BC is the same as that of BD to DE by *Sup.* it follows by 11. 5. that the Ratio of BD to BG, is equal to that of BD to DE, and by 14. 5. the Side DE is equal to the Side DG: After the same manner the Ratio of AB to AC is the same as that of BD to BG, and since the Ratio of AB to AC is suppos'd the same as that of BD to BE, the Ratio of BD to BG will be similar to that of BD to BE, and the Side BG, will be equal to the Side BE; consequently by 8. 1. the Triangle BDE will be equiangular to the Triangle BDG, and consequently to the Triangle ABC. *Which was to be demonstrated.*

USE.

The Method taught in *Prob.* 16. *Introd.* to take an accessible Plan on the Ground, is founded upon this Proposition, very much resembling the eighth of the first Book, that serves also for the Demonstration of this, as has been shewn; for since by 8. 1. if two Triangles have their Sides equal, they themselves are also equal and equiangular, by the same, if the Sides of the two Triangles are proportional, they themselves also are equiangular, consequently by *Def.* they are also similar.

PROPOSITION VI.

THEOREM VI.

Triangles having their Sides about an equal Angle proportional, are equiangular.

Fig. 1. I Say, if the Angle A, of the Triangle ABC, be equal to the Angle B of the Triangle BDE, and the two Sides AB, AC, proportional to these two BD, BE, the Triangle ABC, is equiangular with the Triangle BDE.

PRE-

PREPARATION.

Plate 1, Fig. 1.

Make at the Extremity B, of the Side BD, by 23. 1. an Angle DBG equal to the Angle A, or DBE supposed equal to the Angle A, and at the other Extremity D, the Angle BDG equal to the Angle ABC.

DEMONSTRATION.

Because the Triangles ABC, BGD are equiangular by *Constr*. the Ratio of the two Sides AB, AC, will be equal to that of the two BD, BG, by *Prop*. 4. and because the Ratio of the same two Sides AB, AC, is also equal to that of the two BD, BE, by *Sup*. it follows by 11. 5. that the Ratio of BD to BG, is equal to that of BD to BE, and by 14. 5. that the Side BG is equal to the Side BE: wherefore by 4. 1. the Triangle BDF will be equiangular with the Triangle BDG, and consequently with the Triangle ABC. *Which was to be demonstrated.*

USE.

The Demonstration of *Prop*. 20. depends upon this, which very much resembles the fourth of the first Book, used in the Demonstration of this; for since by 4. 1. two Triangles having two Sides, and the Angle contained equal, are in all respect equal and equiangular, by the same two Triangles having two Sides proportional, and the Angle contain'd equal, are also equiangular, and consequently by *Prop*. 4. they are similar.

Prop. VII. *is needless.*

PROPOSITION VIII.

Plate I. Fig. 7.

THEOREM VIII.

A Perpendicular let fall from the Right-Angle of a right-angled Triangle upon the opposite Side, divides the Triangle into two others similar to it self.

I Say, if you let fall a Perpendicular DA, to the opposite Side BC, call'd the Hypotenuse, from the Right-Angle D, of the right-angled Triangle BDC, each of these two right-angled Triangles DAB, DAC, will be similar to BDC the Triangle proposed; so that the Angle ADC will be equal to the Angle B, and the Angle ADB equal to the Angle C.

DEMONSTRATION.

Because the Angle A of the Triangle ADB is right, by *Sup.* the Sum of the two others B, ADB, will also by 32. 1. be right, and consequently equal to the Angle BDC, which is right by *Sup.* Wherefore taking away the common Angle ADB, there will remain the Angle B equal to the Angle ADC: So also because the Angle A, of the Triangle ACD is right, the Sum of the two others C, ADC is equal to a right one also, that is to say, to the Angle BDC, consequently take away the Angle ADC, and you will have the Angle C, equal to the Angle ADB. *Which was to be demonstrated.*

USE.

This Proposition serves to find a Mean proportional between two Lines given, as shall be shown in *Prop.* 15. because the Perpendiculas AD, is a Mean proportional between the two Parts or Segments AB, AC, the Triangles ADB, ADC, being similar; consequently by *Prop.* 4. the two Sides AB, AD, of the Triangle ABD, are proportional to the two AD, AC, of the Triangle ADC: From hence an easy Method of measuring any Right-Line accessible only at one Extremity, by the help of a Square; suppose AC, accessible at the Extremity A, where erect at Right-Angles a Stick AD of a known

known Length, and put the Right-Angle of the Square at the Point D, so as that looking along one of its Sides DC, you may perceive the Point C, and along the other DB another Point, as B, then since the Lines AB, AD, AC, are proportional, multiply the Length of the Stick AD by it self, and divide the Product by the Quantity of the Line AB, and you will have that of the Line AC sought. Plate 1. Fig. 7

PROPOSITION IX.

PROBLEM I.

To cut off any Part of a given Line.

TO cut off, for instance, a third Part from the given Line AD, draw the Line AE at pleasure, and having taken the Line AC of an arbitrary Length, take AE tripple the Line AC, and draw thro' the Point C, the Line BC, parallel to the Line DE, and that will cut off the Line AB, equal to a third Part of the Line AD proposed. Fig. 8.

DEMONSTRATION.

Because the two Lines BC, DE, are parallel, the Angle ABC will be equal to the Angle ADE, by 29 1. and because the Angle A is common, the Triangle ABC will be equiangular to the Triangle ADE, by 32. 1. Wherefore by *Prop.* 4. the Ratio of the Lines AE, AC will be equal to that of the Lines AD, AB; and since AE is triple AC, by *Const.* AD also will be triple AB. *Which was to be demonstrated.*

USE.

This Proposition serves to divide a given Line into as many equal Parts as you please; for 'tis plain, that to divide the Line AD, into three equal Parts, for instance, no more is necessary than to cut off a third Part AB, as has been shewn.

PROPOSITION X.

PROBLEM II.

To divide a given Line in the same manner as another given Line is divided.

Plate 1. Fig. 8. TO divide the given Line AD at the Point B, just as the Line AE is divided in C, so that the Ratio of the two Parts AB, BD, be equal to that of AC, CE; join the two given Lines AD, AE, at any Angle you please, as DAE, and having joined the Right-Line DE, draw the Right-Line BC, parallel to the Line DE, thro' the Point C, and the two Parts AB, BD, will be proportional to those two AC, CE.

DEMONSTRATION.

Because the Line BC is parallel to the Side DE of the Triangle ADE, by *Const.* the Ratio of the two Parts AB, BD, will be by *Prop.* 2. equal to that of AC, CE. *Which was to be demonstrated.*

USE.

This Proposition may be very well used in dividing a given Line into as many equal Parts as you please; for 'tis evident that if the two Parts AC, CE, were equal, AB, BD, would also be equal. *See Prob.* 14. *Introd.*

PROPOSITION XI.

PROBLEM III.

To find a third Line proportional to two given Lines.

Fig. 9. TO find a third Line proportional to the two Lines AB, AC, make any Angle BAC, with the two given Lines, and applying the Length of the second Line

given

given AC to the first AB, from A to CE, join the Right-Line BC, and draw ED parallel to it, and the Line AD will be the third proportional to the two given Lines AB, AC. Plate 1. Fig. 9.

DEMONSTRATION.

Because the two Triangles ABC, ACD are equiangular, as you have seen in *Prop.* 9. the Ratio of the two Sides AB, AC, of the Triangle ABC, will be like that of the two Sides AE, AD, of the Triangle AED, by *Prop.* 4. So that the Line AD will be a third Proportional to the two AB, AC. *Which was to be demonstrated.*

USE.

This Proposition may be used in reducing a given Square into a Rectangle of a given Height; by finding a third Proportional to the Height sought, and the Side of the given Square, and that will be the Base of the Rectangle sought, as is evident from *Prop.* 17. This Proposition is also used in the Demonstration of *Prop.* 19.

PROPOSITION XII.

PROBLEM IV.

To find a fourth Proportional to three given Lines.

TO find a fourth Proportional to the three given Lines, AB, AC, AD, make any Angle BAC with the two former, AB, AC, and joining the Right-Line BC, apply the Length of the third given Line AD, to the first AB, from A to D; and draw from the Point D a Line DE parallel to the Line BC, thro' the Point D, and the Line AE will be a fourth Proportional to the three Lines given AB, AC, AD. Fig. 8.

DEMONSTRATION.

Because the Line BC, is parallel to the Line DE, by

Plate 1. Fig. 8. *Const.* the Triangle ABC will be equiangular with the Triangle ADE, as we saw in *Prop.* 9. Consequently by *Prop.* 4. the four Lines AB, AC, AD. AE, will be proportional. *Which was to be demonstrated.*

USE.

This Proposition serves to reduce a given Triangle into another of a given Height, by finding a fourth Proportional to the given Height, and the two Sides of the given Rectangle, and that will be the Base of the Rectangle sought, as is plain by *Prop.* 16.

PROPOSITION XIII.

PROBLEM V.

To find a Mean proportional between two given Lines.

Fig. 7. TO find a Mean proportional between the two given Lines AB, AC, form one Right-Line BC out of them both, and describe the Semicirle ADC upon it, and erect from A, a Perpendicular AD upon the Line BC, and that will be a Mean proportional between AB, AC.

DEMONSTRATION.

Join the Right-Lines BD, CD, and by 31. 3. you will find the Angle BDC is right, and by *Prop.* 8. the Line AD is a Mean proportional between AB, AD. *Which was to be effected and demonstrated.*

SCHOLIUM.

If the Paper be not long enough to form a Right-Line out of the two proposed AB, AC, cut off from the greatest AC, the Part AE, equal to the least AB, and having describ'd upon AC, the Semicircle AFC, draw from the Point E, the Right-Line EF perpendicular to the same Line AC, and join the Right-Line AF, and it will be a Mean proportional between the two Lines proposed AB, AC.

DE-

DEMONSTRATION.

Join the Right-Line CF, and by 31. 3. you will find the Angle AFC is right, and by *Prop.* 8. the two Right-angled Triangles FEA, FEC, are equiangular to the great one AFC; consequently by *Prop.* 4. the Ratio of the two Sides AC, AF, of the Triangle AFC, is equal to that of the two Sides AF, AE, of the Triangle AEF, wherefore the Line AF is a Mean proportional between AC and AE, or AB, its equal. *Which was to be demonstrated.* See *Prop.* 17. Plate 1. Fig. 7.

USE.

As the former Proposition serves to do the Rule of Three, so this serves to find in Lines the Square Root of a Number proposed, namely, by finding a Mean proportional between the Number proposed and Unity, for that will be the Root sought, by *Prop.* 17.

PROPOSITION XIV.

THEOREM IX.

Equiangular and equal Parallelograms are reciprocal, and Reciprocal Parallelograms are equiangular and equal.

I Say, first, if the Parallelograms ACD, ABE, are equiangular and equal, they are also reciprocal, that is to say, the Side AC is to the Side AB, as the Side AE to the Side AD. Fig. 2.

PREPARATION.

Imagining the two Parallelograms ACD, ABE, so plac'd as that the Sides AB, AC, may be in a Right-Line, in which Case the two other Sides AD, AE, will also be a Right-Line, by 14. 1. Because the Angle CAD is equal to the Angle BAE, by *Sup.* Produce the other Sides till they intersect in F, and form the Parallelogram AF.

DE-

Plate 1. Fig. 2.

DEMONSTRATION.

Becauſe the Parallelograms CD, BE are equal by *Sup.* they have the ſame Ratio to the Parallelogram AF by 7. 5. and becauſe by *Prop.* 1. the Parallelogram CD is to the Parallelogram AF, as the Baſe AC to the Baſe AB, and the Parallelogram BE is alſo to the Parallelogram BD, as the Baſe AE to the Baſe AD, it follows that the Ratio of the two Lines AC, AB, is equal to that of AE, AD. *Which was to be demonſtrated.*

I ſay, in the ſecond Place, that if the Parallelograms ACD, ABE, are equiangular and reciprocal, they are alſo equal.

DEMONSTRATION.

If a Conſtruction be made like to the foregoing, by *Prop.* 1. Since the Ratio of AC to AB is equal to that of AE to AD, by *Sup.* The Ratio alſo of the Parallelogram ACD, to the Parallelogram AF, is equal to that of the Parallelogram ABE, to the ſame Parallelogram AF, and by 9. 5. the two Parallelograms ACD, ABE are equal. *Which remain'd to be demonſtrated.*

USE.

This Propoſition ſerves to demonſtrate *Prop.* 16. and that Rule in Arithmetic call'd *The Rule of Three inverſe.*

PROPOSITION XV.

THEOREM X.

The equal Triangles, that have one Angle equal, have the Sides about that equal Angle reciprocally proportional; and if the Sides are reciprocally proportional, the Triangles are equal.

Fig. 3.

I Say, firſt, if two Triangles ABC, DBE, are equal, and the Angle ABC equal to the Angle EBD, the Ratio of the two Sides AB, BD, is equal to that of BE, BC.

PRE-

PLATE I. Fig. 3.

PREPARATION.

Imagine the two Triangles ABC, EBD, plac'd so as that the two Sides AB, BD, be in a Right-Line, in which Case BE, and BC will also form a Right-Line, by 14. 1. Because the Angle ABC, is equal to the Angle DBE, by *Sup*. and join the Right-Line AE.

DEMONSTRATION.

Because the Triangles ABC, EBD are equal, by *Sup*. they will have the same Ratio to the Triangle ABE, by 7. 5. and because by *Prop*. 1. the Triangle ABE is to the Triangle BED, as the Base AB is to the Base BD, and in like manner the Triangle ABE is to the Triangle ABC, as the Base BE, to the Base BC, it follows that the four Lines AB, BD, BE, BC are proportional. *Which was to be demonstrated.*

I say, in the second Place, if the two Angles ABC, EBD, are equal, and the Sides AB, BD, BE, BC, proportional, the Triangles ABC, EBD are also equal.

DEMONSTRATION.

Make a Construction like to the preceding, and by *Prop*. 1. since the Ratio of AB to BD, is equal to that of BE to BC; by *Sup*. The Ratio also of the Triangle ABE, to the Triangle EBD, is similar to that of the Triangle ABE, to the Triangle ABC, and by 14. 5. the two Triangles ABC, EBD are equal. *Which remain'd to be demonstrated.*

USE.

This Proposition serves to demonstrate *Prop*. 19. and *that two Right-Lines intersect one another proportionally between Parallels*, because if you join the Right-Line CD, it will be parallel to the Right-Line AE, by 39. 1. the Triangle ACD being equal to the Triangle, CED, *&c.*

PRO-

PROPOSITION XVI.

THEOREM XI.

If four Lines are proportional, the Rectangle of the two Extreams is equal to the Rectangle of the two Means; and if the Rectangle of the two Extreams be equal to that of the two Means, the four Lines are proportional.

Plate 1. Fig. 2.

I Say, first, if the four Lines AB, AC, AD, AE, are proportional, the Rectangle ABE, of the Extreams AB, AE, is equal to the Rectangle of the Means AC, AD.

DEMONSTRATION.

Because the four Lines AB, AC, AD, AE, are proportional, by *Sup.* the Rectangles ABE, ACD will be reciprocal, by *Def.* 2. and since they are equiangular, by *Const.* it follows from *Prop.* 15. that they are equal. *Which was to be demonstrated.*

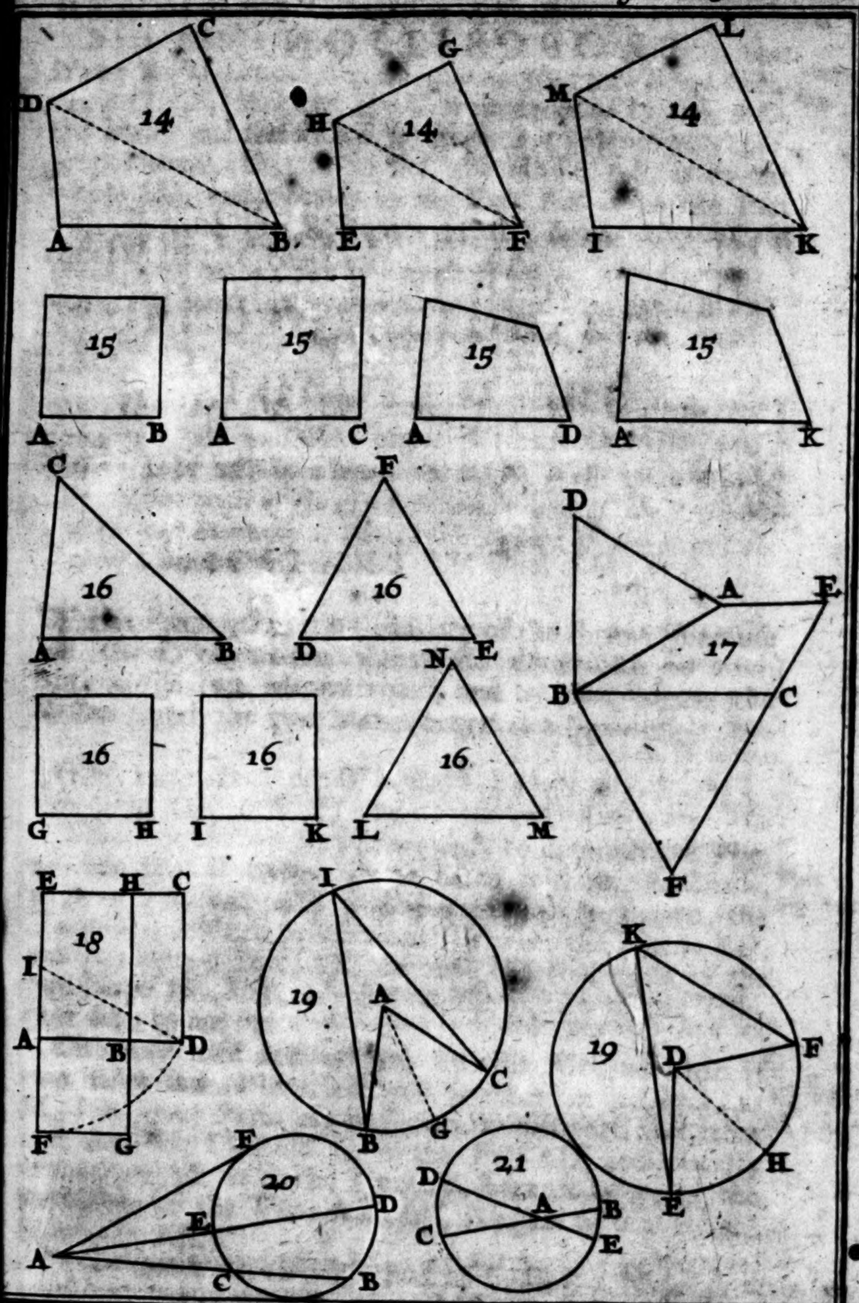
I say, in the second Place, if the Rectangles ACD, ABE, are equal, the four Lines AB, AC, AD, AE, are proportional.

DEMONSTRATION.

Because the two Rectangles ACD, ABE, are equal by *Sup.* and equiangular by *Const.* they are reciprocal by *Prop.* 14. that is to say by *Def.* 2. the four Lines AB, AC, AD, AE, are proportional. *Which was what remain'd to be demonstrated.*

USE.

This Proposition serves to demonstrate the Rule of Three, because the Area of a Rectangle being found by multiplying the two Sides that form the Right-Angle together, as has been seen in the second Book, 'tis easy to conclude from this Proposition, that in four proportional Quantities, the Product of the two Extreams is equal



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equal to that of the two Means; and so on the contrary. Which we have already demonstrated. Plate 1. Fig. 2.

It may also be demonstrated by this Proposition, that if two Right-Lines intersect one another in a Point without a Circle, and cut the Circumference, as AB, AD, the whole and their external Parts are reciprocally proproportional, that is to say, the whole AB, is to the whole AD, reciprocally as the Part AE is to the Part AC, because the Rectangle of the Lines AB, AC, is equal to that of the Lines AD, AE. Plate 2. Fig. 20.

PROPOSITION XVII.

THEOREM XII.

If three Lines are proportional, the Square of the Mean is equal to the Rectangle of the two Extreams; and if the Rectangle of the two Extreams is equal to the Square of the Mean, the three Lines are proportional.

THis Proposition is a Corollary of the former, because three proportional Lines are equivalent to four, having the two Means equal, and by that Means the Rectangle of the two Means becomes a Square.

USE.

This Proposition serves not only to demonstrate *Prop.* 30. but that if from a Point taken without the Circle, as A, a Tangent AF, and Secant AD be drawn, the Tangent is a Mean proportional between the Secant AD, and its external Part AE, because the Rectangle of the two Lines AD, AE, is equal to the Square of the Tangent AE, by 36. 3. Plate 2. Fig. 20.

One may also demonstrate by this Method, that if two Right-Lines intersect one another in a Circle, as BC, DE, their Parts are reciprocally proportional, that is to say, the Part AB, is to the Part AD, reciprocally as the Part AE is to the Part AC, because by 35. 3. the Rectangle of the Parts AB, AC, is equal to that of the Parts AD, AE. Fig. 21.

From hence an easy Method of finding a Mean proportional between two given Lines, as AD, AE, may be drawn, namely, describing on the Difference DE, a Circumference of a Circle, and drawing the Tangent AF, which will be the mean proportional sought. Fig. 20.

PRO-

PROPOSITION XVIII.

PROBLEM VI.

To describe upon a given Line a Polygon similar to a given one.

Plate I. Fig. 10.

TO describe on the Line EF, a Polygon similar to the given one ABCD, draw the Diagonal BD, and the Angle E being made equal to the Angle A, make also the Angle EFH equal to the Angle ABD. Make the Angle FHG equal to the Angle BDC, and the Angle HFG equal to the Angle DBC, and the Figure EFGH will be similar to the proposed one ABCD, that is to say, all the Angles of the one, will be equal to all the Angles of the other, and the Sides proportional.

DEMONSTRATION.

'Tis already evident by *Const.* that the two Polygons ABCD, EFGH are equiangular, because all the Triangles of the Polygon ABCD are made equiangular with all the Triangles of the Polygon EFGH, so that all that remains, is to demonstrate that the Sides are proportional.

Because the three Triangles ABD, EFH, are equiangular by *Const.* it follows by *Prop.* 4. that the two Sides AB, AD, are proportional to EF, EH; and so also because the two Triangles BCD, FGH, are equiangular, the two Sides BC, CD are proportional to those two FG, GH. But I say further, the two Sides AB, BC, are also proportional to the two EF, FG, and the two AD, CD, to the two EH, GH, as we shall now demonstrated.

Because in the two equiangular Triangles ABD, EFH, the Ratio of the two Sides AB, BD, is like that of the two EF, FH, by *Prop.* 4. and in like manner in the equiangular Triangles BCD, FGH, the Ratio of the two Sides BD, BC, is equal to that of the two FH, FG; so that the three Lines BA, BD, BC, are Proportional to the three Lines FE, FH, FG, and by

by 22. 5. the Ratio of the two Sides AB, BC, is like that of the two EF, FG. *Which is one of the things that was to be demonstrated.* Plate I. Fig. 10.

After the same manner in the two equiangular Triangles ABD, EFH, the Ratio of the two Sides AD, BD, is equal to that of the two EH, FH; and in like manner in the two equiangular Triangles BCD, FGH, the Ratio of the two Sides BD, CD is the same with that of the two FH, GH. Thus you see that the three Lines DA, DB, DC, also proportional to the three Lines HE, HF, HG, and by 22. 5. the Ratio of the two Sides AD, CD, is equal to that of the two EH, GH. *Which is what remain'd to be demonstrated.*

USE.

This Proposition is the Foundation of what is taught in *Prob.* 17. *Introd.* to take an inaccessible Plan on the Ground; as also of the Method ordinarily used to trace upon the Ground the Plan of a Fortress, whose Design is drawn upon Paper: for since you can't work it upon the Ground as upon Paper, you must make upon the Ground Angles equal to those of the Plan described on Paper.

PROPOSITION XIX.

THEOREM XIII.

Equiangular Triangles are in a Duplicate Ratio of that of their Homologous Sides.

Homologous Sides are the Sides of two similar Rectilineal Figures, that are opposite to the equal Angles: Thus if the two Triangles ABC, DEF, are equiangular, and consequently similar, by *Prop.* 4. so that the Angle A is equal to the Angle D, and the Angle B to the Angle E, and consequently the third Angle C equal to the third Angle F; the two Sides AB, DE, that are opposite to the two equal Angles C, F, are *Homologous*. Fig. 12.

This being supposed, I say the Ratio of the two Triangles ABC, DEF, is the Duplicate of that of the two

Plate I. Fig. 12. Homologous Sides AB, DE, that is to say, if by *Prop.* 11. you find a third proportional Line AG, to the two Homologous Sides AB, DE, the Triangle ABC is to the Triangle DEF as the first proportional AB is to the third proportional AG.

DEMONSTRATION.

Because the Triangles ABC, DEF, are equiangular, by *Sup.* the Ratio of the two Sides AC, DF, is equal to that of the two AB, DE, which is also equal to that of DE, AG, by *Const.* because the Line AG was made a third proportional to AB, DE: consequently by 11. 5. the Ratio of the two Sides AC, DF, will be equal to that of DE, AG, and the Angle A being equal to the Angle D, by *Sup.* the Triangle ACG, will be equal to the Triangle DEF, by *Prop.* 15. and since the Triangle ABC is to the Triangle AGC, as the Base AB to the Base AG, by *Prop.* 1. the Triangle ABC is to the Triangle DEF, as the first Proportional AB, to the third Proportional AG. *Which was to be demonstrated.*

COROLLARY.

It follows from this Proposition, that equiangular Triangles are as the Squares of their Homologous Sides; since the Triangle here ABC, is to the Triangle DEF, as the Square of the Side AB, namely AI, to the Square DL of the Homologous Side DE, because these two Squares are to one another as their halves, by 15. 5. and consequently as the Triangles ABH, DEK, which being equiangular by 4. 2. are in a Duplicate Ratio of their Homologous Sides AB, DE, as the Triangles ABC, DEF.

USE.

This Proposition serves to undeceive such as easily imagine that similar Figures are as their Sides, since it is certain if the Sides of the one for instance, are double the Sides of the other, the greater will be Quadruple the less, because the Duplicate Ratio of the Double is the Quadruple.

PRO-

PROPOSITION XX.

THEOREM XIV.

Similar Polygons may be divided into as many similar Triangles; and similar Polygons are in the Duplicate Ratio of their Sides.

I Say first, if the Polygons ABCDE, FGHIK, are similar, they may be divided into as many similar Triangles that will be similar Parts of their Polygons, each of its own. Plate 1. Fig. 13.

DEMONSTRATION.

Draw the Diagonals DA, DB, IF, IG; and by *Prop.* 6. the two Triangles AED, FKI, are similar, because the Angles E, K are equal, and the Sides EA, ED, are proportional to KF, KI, the two Polygons proposed being supposed similar. And so also you may find that the Triangle BCD is similar to the Triangle GHI. Consequently 'tis easy to conclude that the two other Triangles ADB, FIG are also similar, because equiangular. *Which was to be demonstrated.*

I say, in the second Place, the similar Polygons ABCDE, FGHIK, are in a Duplicate Ratio of their Homologous Sides.

DEMONSTRATION.

Since the two Polygons are made up of similar Triangles, as has been demonstrated, and they are all in a Duplicate Ratio of their Homologous Sides, by *Prop.* 19. and the Ratio of the Sides is the same, the Polygons being supposed similar, the Duplicate Ratio will also be the same, and so each Triangle of one Polygon will be to each Triangle of the other in the same Ratio, and by 12. 5. the Ratio of each Triangle to its similar, will be the

Plate 1. Fig. 13. the ſame with that of the Sum of all the Triangles of one Polygon, to the Sum of all the Triangles of the other; that is to ſay, of one Polygon to the other: and becauſe the Ratio of theſe two Triangles is the Duplicate of that of their Homologous Sides, the Polygon alſo muſt be in the Duplicate Ratio of that of their Homologous Sides. *Which was to be demonſtrated.*

COROLLARY.

From this Propoſition it follows, that ſimilar Polygons are as the Squares of their Homologous Sides; and that three Lines being proportional, the Polygon deſcrib'd upon the firſt, is to the ſimilar Polygon deſcrib'd upon the ſecond, as the firſt Line is to the third, becauſe that Ratio is the Duplicate of that of the firſt to the ſecond, that are two Homologous Sides of theſe two Polygons.

USE.

This Propoſition is of uſe in *Prop.* 21. and 22. and to encreaſe a given Polygon in a given Ratio; as if you would have a Polygon quadruple of another, double all the Sides, for the Duplicate Ratio of the double is quadruple; and ſo if you would have a Polygon noncuple of another, triple all the Sides, becauſe the Duplicate of the Triple is Noncuple.

But 'tis evident that to leſſen a given Polygon according to a given Ratio, the contrary is to be done; ſo that if you would have a Polygon but a quarter of that propoſed, you muſt take half the Sides.

And if any other Ratio were propoſed, for inſtance, that of 2 to 3, find a Mean proportional between the double of one Side of the Polygon propoſed and its Triple, and that will be the Homologous Side of the Polygon ſought.

PROPOSITION XXI.

THEOREM XV.

Two Polygons ſimilar to a third, are ſimilar to one another.

Plate 2. Fig. 14. I Say, if each of the two Polygons ABCD, IKLM is ſimilar to the Polygon EFGH, theſe two Polygons ABCD, IKLM, are ſimilar to each other.

DE-

DEMONSTRATION.

Plate 2. Fig. 14.

Because the Polygons ABCD, EFGH are ſimilar, by *Sup.* one may be divided by Diagonals into as many ſimilar Triangles as the other, by *Prop.* 20. as here into two, the Triangle ABD, being ſimilar to the Triangle IKM, and the Triangle BCD, to the Triangle FGH. Thus alſo the Polygon IKLM being ſuppoſed ſimilar to the Polygon EFGH, the Triangle IKM will be ſimilar to the Triangle EFH, and conſequently to the Triangle ABD, becauſe two Angles equal to a third, are equal to one another; and ſo alſo the Triangle KLM will be ſimilar to the Triangle FGH, and conſequently to the Triangle BCD. Conſequently the Polygons ABCD, EFGH being compoſed of an equal Number of equiangular Triangles, will alſo be equiangular, becauſe their ſimilar Triangles having their reſpective Angles equal, the Angles of the Polygon made up of them will alſo be equal; and becauſe theſe ſimilar Triangles have their Sides proportional, by *Prop.* 4. the Polygons alſo will have their Sides proportional, and by *Def.* 1. will be ſimilar. *Which was to be demonſtrated.*

PROPOSITION XXII.

THEOREM XVI.

If four Right-Lines are proportional, the ſimilar Polygons deſcribed on thoſe Lines, will alſo be proportional; and if they are proportional, the four Lines will alſo be proportional.

I Say, firſt, if the four Lines AB, AC, AD, AE, are proportional, the four ſimilar Polygons form'd upon thoſe Lines, for inſtance, the two Squares and two Trapeziums, will be proportional. Fig. 15.

Plate 2. Fig. 15.

DEMONSTRATION.

Because the four Lines AB, AC, AD, AE, are proportional, by *Sup.* the Duplicate Ratio of the two first, AB, AC, is equal to the Duplicate Ratio of the two last AD, AE; and since by *Prop.* 20. the Duplicate Ratio of the two first AB, AC, is equal to that of their similar Polygons, and the Duplicate Ratio of the two last AD, AE, is equal to that of their similar Polygons, it follows, that these four Polygons are proportional. *Which was to be demonstrated.*

I say, in the second Place, if four similar Polygons form'd on the four Lines AB, AC, AD, AE, are proportional, these four Lines will also be proportional.

DEMONSTRATION,

Because the Ratio of the two first Polygons is equal to that of the two last, by *Sup.* and each is the Duplicate of that of their Homologous Sides, by *Prop.* 20. the four Homologous Sides and consequently the four Lines AB, AC, AD, AE, are proportional. *Which remain'd to be demonstrated.*

USE.

This Proposition serves to do the Rule of Three Geometrically, when three Figures being given, a fourth Proportional is to be found, namely by reducing the three Figures proposed into three Squares, when they are not similar, and finding a fourth Proportional to the Sides of the three Squares, and that will be the Side of a Square equal to the fourth Proportional Figure sought. This Proposition serves also to demonstrate *Prop.* 1. 11.

PRO-

PROPOSITION XXIII.

THEOREM XVII.

Equiangular Parallelograms are in a Ratio compounded of that of their Sides.

I Say, if the two Parallelograms ACD, ABE, are equiangular, their Ratio is compounded of the Ratio of the Side AC, to the Side AB, and of the Ratio of the Side AD, to the Side AE. Plate I. Fig. 2.

PREPARATION.

Having imagin'd the two Parallelograms ACD, ABE, placed so as that the Sides AB, AC may be in a Right-Line, in which Case the two other Sides AD, AE, will also be in a Right-Line, by 14. 1. because the Angle CAD, is equal to the Angle BAE; produce the other Sides till they meet in a Point, as F, and so make a third Parallelogram AF.

DEMONSTRATION.

Because in the three Parallelograms ACD, AF, ABE, the Ratio of the first to the third is composed of the Ratio of the first to the second, which is equal to that of the Base AC to the Base AB, and of the Ratio of the second to the third, which is also equal to that of the Base AD to the Base AE; it follows that the Ratio of the Parallelogram ACD, to the Parallelogram ABE, is composed of the Ratio of the Side AC to the Side AB, and of the Ratio of the Side AD to the Side AE. *Which was to be demonstrated.*

SCHOLIUM.

Plate I. Fig. 2.

If you would compound the Ratio's of AC to AB, and of AD to AE, you must multiply the two Antecedents AC, AD together, and so you will have the Content of the Parallelogram ACD; multiply also the two Consequents AB, AE, and then you will have the Area of the Parallelogram ABE, in Measures similar to that of the Parallelogram ACD; which is an additional Proof of the two Parallelograms, being in a Ratio compounded of that of their Sides.

Since a Triangle is equal to half a Parallelogram of the same Base and Height, you may easily find by this Proposition, that two Triangles having one Angle equal, are in a Ratio compounded of the Sides that form the Angle, as if they were Parallelograms, which may be easily seen, by drawing the two Diagonals CD, BE, *&c.*

PROPOSITION XXIV.

THEOREM XVIII.

If you draw two Lines parallel to two Sides of a Parallelogram, thro' a Point in the Diagonal, there will be formed four Parallelograms, of which those two that the Diagonal passes thro', are similar to one another and to the great one.

Plate I. Fig. 11.

I Say, if thro' the Point E taken at Discretion in the Diagonal BD of the Parallelogram ABCD, you draw the two Lines FG, HI, parallel to the two Sides AD, AB, the two Parallelograms GH, FI, are similar to one another and to the great one.

DEMONSTRATION.

Because the Line HI is parallel to AB, by *Sup.* the Angle DHE will be equal to the Angle A, by 29. 1. which makes the two Triangles DHE, DAB similar: Consequently by *Prop.* 4. the Ratio of DH to HE, will be equal to that of AD to AB, and by *Def.* 1. the Parallelogram GH will be similar to the Parallelogram ABCD. After the same manner you may find that the Parallelogram

gram FI is similar to the same Parallelogram ABCD, and consequently to the Parallelogram GH. *Which was to be demonstrated.* Plate 1. Fig. 11.

SCHOLIUM.

The Converse of this Proposition is also certainly true, namely, that if the Parallelogram GH, or FI, be similar to the great one ABCD, having an Angle common, the Diagonal of the great one drawn thro' the common Angle, will pass thro' the other Angle of the less, as *Euclid* has demonstrated in *Prop.* 26. which we omit, because easily understood, and of little Use.

PROPOSITION XXV.

PROBLEM VII.

Two Rectilineal Figures being given, to describe a third equal to one of the given ones, and similar to the other.

TO describe a Rectilineal Figure equal to the given one ABC, and similar to the given one DEF, reduce into a Square each of the two Rectilineal Figures given, ABC, DEF, by 14. 2. So that GH be the Side of a Square equal to the Rectilineal Figure ABC, and IK the Side of a Square equal to the Rectilineal Figure DEF. Then find by *Prop.* 12. a fourth Proportional LM, to the three Lines IK, GH, DE, and by *Prop.* 18. describe upon that Line LM, the Rectilineal Figure LMN, similar to the Rectilineal Figure DEF, which here is an equilateral Triangle, and the Rectilineal Figure LMN, will be equal to the Rectilineal Figure ABC. Plate 2. Fig. 16.

DEMONSTRATION.

Because the four Lines IK, GH, DE, LM, are proportional, by *Constr.* their Squares will also be proportional, by *Prop.* 22. and because the Squares of the two Lines DE, LM, are in the same Ratio as the two similar Rectilineal Figures DEF, LMN, by *Prop.* 20. the Ratio of the Squares of those two Lines IK, GH, is equal to that of the two Rectilineal Figures DEF, LMN; and since

Plate 2. Fig. 26. since the Square of the Line IK, is equal to the Rectilineal Figure DEF, by *Constr*. Then by 14. 5. the Square of the Line GH, or the Rectilineal Figure ABC, is equal to the Rectilineal Figure LMN. *Which was to be effected and demonstrated.*

USE.

The use of this Proposition is more extensive than that of *Prop*. 14. 2. by which the Rectilineal proposed can only be reduced into a Square, whereas this Proposition serves to reduce it into any other Figure you please; thus here we have reduced the Scalene Triangle ABC, into an equilateral Triangle. We have resolved this Problem otherwise than *Euclid* has, because his Method depends on a Proposition in the first Book, that we have omitted because it seem'd too perplex'd.

We shall here omit Prop. XXVI. XXVII. XXVIII. *and* XXIX. *that are but of little Consequence.*

PROPOSITION XXX.

PROBLEM X.

To cut a Right-Line in extream and mean Proportion.

Fig. 18. TO divide the given Right-Line AD, into extream and mean Proportion, cut it at the Point B, by 11. 2. So that the Rectangle of the whole AD, and its lesser Part BD, namely the Rectangle BC, be equal to the Square AG, of the greater Part AB, and the Problem is solved.

DEMONSRATION.

Because the Rectangle BC is equal to the Square AG of the Line AB, by *Constr*. the three Lines CD, or AD, AB, BD, will be proportional, by *Prop*. 17. and *Def*. 3. the Line AD will be cut at the Point B, in extream and mean Proportion. *Which was to be effected and demonstrated.*

USE.

USE.

A Line thus cut has several Properties, as may be seen in a Book published by *Lucas de sancto Sepulchro*, and serves, as has been shewn, to describe a Pentagon and a regular Decagon; and *Euclid* uses it in the thirteenth Book, to determine the Sides of the five regular Bodies. Plate 2. Fig. 18.

PROPOSITION XXXI.

THEOREM XXI.

If you describe three similar Rectilineal Figures upon the three Sides of a Rectangle Triangle, that which is form'd upon the Side opposite to the Right-Angle, is equal to the Sum of the two others.

I Say, if you describe upon the Sides of the Triangle ABC, right-angled in A, three similar Rectilineal Figures, for instance, the three Triangles ABD, ACE, BCF, the Triangle BCF, is equal to the Sum of the other two ABD, ACE. Fig. 17.

DEMONSTRATION.

Because by *Prop.* 20. the Rectilineal Figure ABD is to the Rectilineal Figure ACE, as the Square AB, to the Square AC, and compounding by 18. 5. the Sum ABD+ACE, will be to ACE, as the Sum of the two Squares AB, AC, that is to say, by 47. 1. as the Square BC, to the Square AC; and because the Ratio of the Square BC to the Square AC, is equal to that of the Rectilineal Figure BCF, to its similar one ACE, by *Prop.* 20. then by 11. 5. the Ratio of the Rectilineal Figure BCF, to the Rectilineal Figure ACE, is equal to that of the Sum ACD+ACE, to the same Rectilineal Figure ACE, and by 9. 5. the Rectilineal Figure BCF, is equal to the Sum of ACD, ACE. *Which was to be demonstrated.*

USE

USE.

This Proposition serves in general to add several similar Figures together, as we said in 47. 1. so that we need not insist any longer upon it.

We omit Prop. XXXII. *because not necessary, nor of much Consequence.*

PROPOSITION XXXIII.

THEOREM XXIII.

In equal Circles, the Angles at the Center or Circumference, as also their Sectors, are to one another as the Arcs they insist upon.

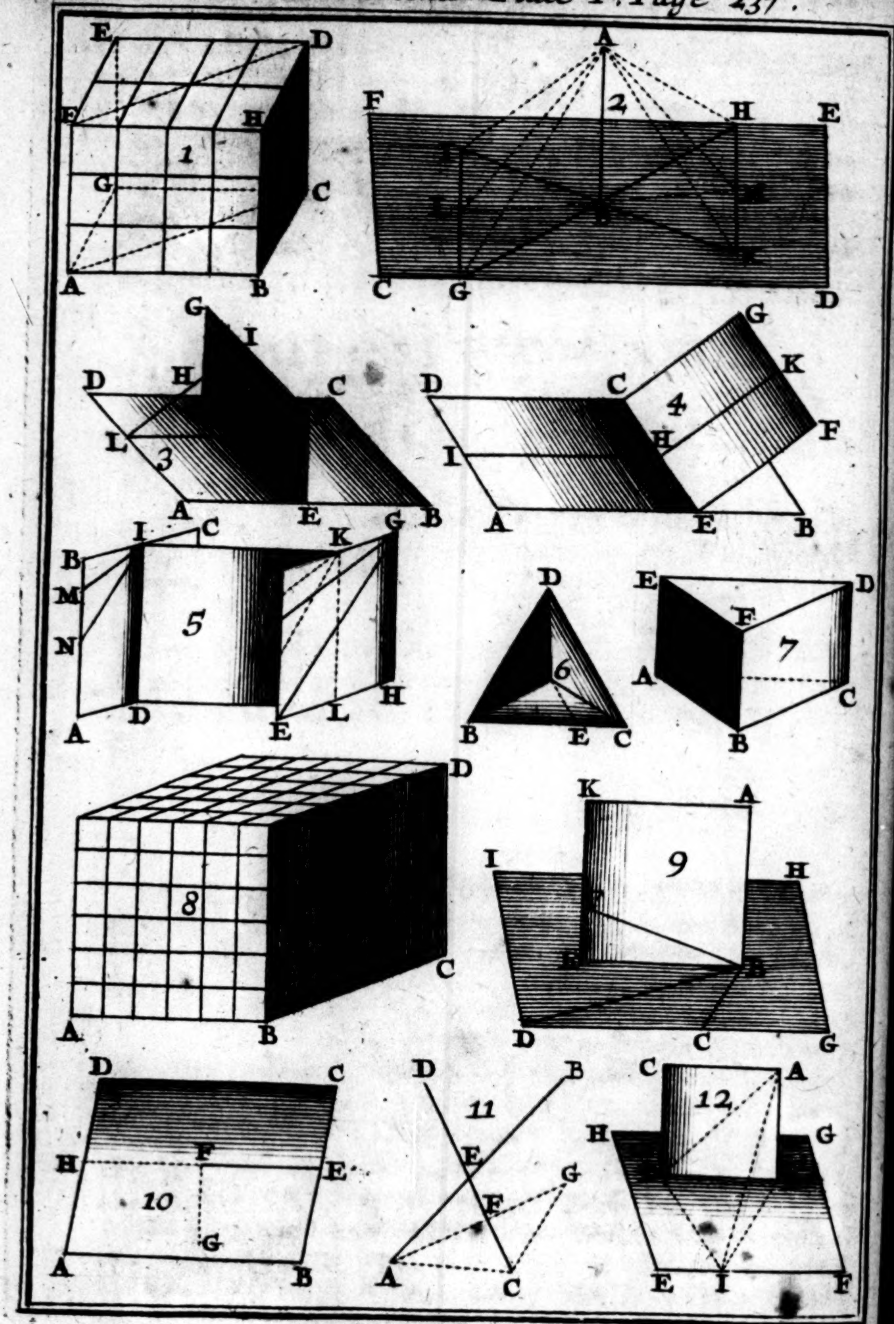
Plate 2. Fig. 19. I Say, first, the two Angles at the Centre BAC, EDF, of the two equal Circles BIC, EKF, are to one another as their Arcs BC, EF, that serve instead of their Base.

PREPARATION.

Bisect each of the two Angles BAC, EDF, with the Radius's AG, DH, and they will bisect the Arcs BC, EF, at the Points G, H, as also the Sectors ABCA, DEFD.

DEMONSTRATION.

Because by 15. 5. the Arc BC is to its half BG, as the Arc EF is to its half EH, and in like manner the Angle BAC, is to its half BAG, as the Angle EDF is to its half EDH, the Proportion between the four Arcs BC, BG, EF, EH, is similar to that between the four Angles BAC, BAG, EDF, EDH; consequently by Conversion, by 16. 5. the Circles BIC, EKF being equal, the Proportion between the four Arcs BC, EF, BG, EH, is similar to that between the four Angles BAC, EDF, BAG, EDH, and consequently in this second Proportion, the Ratio of the first Angle BAC,


1
E D F H G C A B
2
A F H E C G D
3
G I D H C L A E B
4
D C G K F I H A E B
5
I C B K G M N A D E L H
6
D B E C
7
E D F A C B
8
D C A B
9
K A I H E B D C G
10
D C H F E G A B
11
D B E G F A C
12
C A H G E I F

BAC, to the second EDF, is equal to that of the first Plate 1.
Arc BC, to the second EF, in the first Proportion. Fig. 19.
Which was to be demonstrated. Consequently the Angles at the Circumference I, K, being halves of the Angles at the Center A, D, by 20. 3. are also as their Bases BC, EF. After the same manner the Sectors ABCA, DEFD may be demonstrated to be as their Bases BC, EF, considering them as Angles.

SCHOLIUM.

This Demonstration is of the same Nature with that of the first Proposition of this Book; but if the Circles are not equal in this Proposition, or the Heights not equal in the former, you can't reason by Conversion of Proportions.

The

THE ELEVENTH BOOK OF EUCLID's ELEMENTS.

EUclid in this Book begins to treat of a Body or Solid, and first of Parallelopipeds, after he has explain'd in the beginning some Properties of their bounding Surfaces. We omit the seventh, eighth, ninth and tenth Book, because they have no Connexion with the six first, nor with the eleventh and twelfth; we shall only add, because they, and the preceding six, are enough for the tolerable understanding of the principal Parts of Mathematicks; the eleventh and twelfth being absolutely necessary for understanding the third Part of *Practical Geometry*, call'd *Stereometry*, *Spherical Trigonometry*, *Dialling*, *Perspective*, and in general whatever belongs to the Section of Planes and Solids. Such as would have more, may consult *Henrion*, who has demonstrated all the other Books, and the Data.

DEFINITIONS.

I.

Plate I. Fig. 1. A *Body* or *Solid*, is the third Species of Magnitude it has Length, Breadth and Depth. *As ABCD, that has three Dimensions, Length AB, Breadth BC, Depth CD.*

Philoso-

Philosophers divide Bodies into *hard*, or such as do not easily give way to another; and *soft* or such as do, and may easily be penetrated by another. But since the Imagination makes easy and feasible things most difficult in execution, one may imagine a hard Body as easy penetrated as a soft one. And then Mathematicians call a *solid Body*, or a *Solid* simply, whatever is extended in Length, Breadth and Depth, abstracting from Matter, and conceiving a Body produc'd by the Motion of a Surface, as a Surface is by the Motion of a Line, and a Line by the Motion of a Point, and that a Body is made up of an infinite Number of Surfaces, as a Surface is of Lines, and a Line of Points. Consequently,

II.

The Extremities of a Body, are the Surfaces that bound it. Plate I. Fig. I.

A Body is necessarily bounded by Surfaces, as well on the account of what has been said, as because, upon examining a Body as ABCD in particular, you may easily find an Upper Part, namely, the Surface DEF; an Under Part, namely the opposite Surface, ABC, call'd the *Base*; a Fore-part, namely the Surface FAB; a Hinder Part, opposite to that; and Sides, one of which appears in the Figure, represented by the Surface BCD.

III.

A Right-Line is said *to be perpendicular to a Plane, or erected perpendicularly upon a Plane,* that is perpendicular to all the Lines it meets drawn upon the Plane.

Thus the Right-Line AB, is perpendicular to the Plane CDEF, or erected perpendicularly upon it, if it be perpendicular to each of the Lines, GH, IK, LM, that it meets at the Point B, in that Plane. Fig. 2.

IV. One

IV.

Plate 1. *One Plane* is said to be *perpendicular to another, or erected perpendicularly upon another*, when a Right-Line drawn in one of the Planes, perpendicular to their common Section, meets a Perpendicular to the other Plane.

Fig. 3. *Thus the Plane EFGH is perpendicular to the Plane ABCD, or the Plane ABCD to the Plane EFGH, because the Line KL, drawn in the Plane ABCD, perpendicular to the common Section EH; is also perpendicular to the other Plane EFGH: or because the Line IK drawn in the Plane EFGH, perpendicular to the common Section EH, is also perpendicular to the Plane ABCD.*

By the common Section of two Planes, is understood a Line common to those two Planes, in which they intersect, as EH, which always is a Right-Line, as shall be demonstrated in *Prop.* 3.

V.

Fig. 3. *The Inclination of a Right-Line upon a Plane*, is the Acute-Angle made by that Line and another Right-Line, drawn thro' the Point where the Extremity of the Line inclined meets the Plane, and thro' the Point of the same Plane, where it is cut by the perpendicular to that Plane, drawn from the other Extremity of the inclined Line.

Thus the Inclination of the Right-Line IL, with the Plane ABCD, is the Acute-Angle KLI, made with the Line KL drawn thro' the Points L, K, where the Plane ABCD is cut by the inclined Line IL, and the Line IK, perpendicular to the Plane ABCD.

In like manner the Inclination of the same Line IL, to the Plane EFGH, is the Angle KIL, that it forms with the Right-Line IK, drawn thro' the Points I, K, where the Plane EFGH is cut by the inclined Line IL, and the Line LK perpendicular to the Plane EFGH.

VI. The

VI.

Plate 1. Fig. 3.

The Inclination of two Planes is the Acute-Angle of two Right-Lines, perpendicular to the common Section of the two Planes, and drawn thro' the same Point of the same common Section in each Plane.

Thus the Inclination of the two Planes ABCD, EFGH, is the Acute-Angle that the Right-Line HI drawn in the Plane ABCD, perpendicular to the common Section CE, makes with the Line HK, drawn in the Plane EFGC, perpendicular to the same common Section. Fig. 4.

'Tis plain from this Definition, that two Planes must not be perpendicular to each other, that they may be said to be inclined: and from the foregoing Definition that a Right-Line must not be perpendicular to the Plane, that it may be said to be inclined to it.

VII.

Planes similarly inclined are such as have equal Inclinations to a third Plane.

Tho' the Inclination of the Planes, supposes that they are not perpendicular to one another, yet that does not hinder but that two Planes may be said to be similarly inclin'd to a third Plane, when they are perpendicular to it.

VIII.

Parallel Planes are such as being continued as far as you please, will never meet, being always equidistant; *Such are the two Planes ABCD, EFGH, whose Distance IK, DL, perpendicular to them, are equal.* Fig. 5.

IX.

Similar Solids are such as are bounded by an equal Number of similar Planes. *For instance two Cubes.*

X.

Similar and equal Solids, are ſuch as are bounded by an equal Number of ſimilar and equal Planes; ſo that imagining one to penetrate the other, neither would exceed, as having equal Angles and Sides.

XI.

Plate I. Fig. 6. *A Solid Angle* is an indefinite concave Space, terminated in a Point by ſeveral Planes meeting in the Point, where the ſolid Angle is form'd: *As A terminated by the three triangular Planes BAD, CAD, BAC.*

XII.

Fig. 1. *A Priſm*, is a Solid having two oppoſite Planes parallel, ſimilar and equal, and the others Parallelograms: *Thus ABCD, whoſe two oppoſite Planes ABC, DEF, are parallel, ſimilar and equal, and the others, as FAB, BCD, &c. Parallelograms.*

'Tis call'd a *Triangular Priſm*, when its two oppoſite and parallel Planes, are two ſimilar and equal Triangles: Fig. 7. as ABCD, terminated by the three Parallelograms ABFE, ACDE, BCDF, and the two ſimilar parallel and equal Triangles, ABC, EFD.

'Tis call'd a *Parallelopiped*, when 'tis terminated by ſix Parallelograms, of which the two oppoſite and parallel are equal; and when all theſe Parallelograms are Rectangles, the *Priſm* is call'd a *Right-Angled Parallelopiped*, as Fig. 1. ABCD, which take the Name of a *Cube* or *Hexaedrum*, if all its Sides are equal, that is to ſay, when 'tis bounded Fig. 8. by ſix equal Squares, as ABCD, which will repreſent a *Cubic Yard*, if its Side AB be a Yard long: But it will repreſent a Cubic Foot, if the Side AB, BC, or CD, be a Foot long.

We ſaid in the ſecond Book, that the Area of a Rectangle is meaſur'd by little Squares, and we ſhall ſay here that the Content of a Right-Angled Parallelopiped, call'd its Solidity, is meaſur'd by little Cubes, produced by parallel Planes drawn lengthwiſe and croſswiſe, thro' the Diviſions of the oppoſite Sides, which anſwers to the

the Motion of a Surface producing a Solid, and this Motion answers the continual Multiplication according to the three Dimensions of a right-angled Parallelopiped, in finding the *Solidity*, that is to say, the Number of the Cubic Measures it contains. Plate 1. Fig. 1.

Thus the Solidity of the right-angled Parallelopiped ABCD, whose Length AB is here supposed to be 4 Feet, its Breadth BC, 2, and its Depth CD, 3, is found by multiplying these three Numbers 4, 2, 3, together, and the fourth Number that comes forth, namely 24, is call'd *a Solid Number*, whose *Sides* are 4, 2, 3, because they show that a right-angled Parallelopiped, 4 Feet long, 2 Feet broad, and 3 deep, contains 24 Cubic Feet in its Solidity.

Thus because a Yard long, as AB, contains 3 Feet, a Cubic Yard ABCD, will contain 27 Cubic Feet, and from hence 'tis that the Number 27 arising from the mutual multiplication of three equal Numbers, is call'd a Cubic Number, whose *Side*, or *Cube Root* is one of them, namely 3. Fig. 1.

A Rectangled Parallelopiped, in regard of its three Dimensions, is call'd *a Solid of three Lines*, which are its three Dimensions; that is to say, one of these three Lines represents its Breadth, and the other its Length, and the third its Depth, whether the Solid be real or imaginary.

Thus the Solid of the three Lines AB, BC, CD, is the right-angled Parallelopiped ABCD, which is represented in Numbers, when the three Dimensions are expressed by Numbers; as if the Length AB, be 4 Feet, the Breadth BC, 2, and Depth CD, 3, the Solid of these three Numbers 4, 2, 3, will be 24, namely the Product of these three Numbers 4, 2, 3, which on that account is call'd *a Solid Product*, and if you substitute Letters instead of Numbers, as *a*, *b*, *c*, their solid Product will be *abc*. Fig. 1.

The other Definitions belong to the Twelfth Book, and are there explain'd.

PROPOSITION I.

THEOREM I.

A Right-Line in a Plane, if produced, will still be in that Plane.

Plate I. Fig. 10.

I Say, if the Right-Line EF, be in the Plane ABCD, when produced, 'twill still be in the same Plane ABCD.

PREPARATION.

Draw from the Point F, in the Plane ABCD, the Right-Line FG, perpendicular to the Line EF, and another FH, to the Line FG.

DEMONSTRATION.

Because each of the two Angles GFE, GFH, is a right one by *Constr.* the two Lines FH, FE, constitute a right Line, by 14. 1. and because each is in the Plane ABCD, the Line EF produced, that is to say, the whole Right-Line EH, is in the same Plane ABCD. *Which was to be demonstrated.*

USE.

This Proposition serves to demonstrate the following one, and we shall use it in Dialling, to make out that a great Circle of a Sphere is represented on a Plane, by a Right-Line.

PROPOSITION II.

THEOREM II.

Two Right-Lines intersecting one another, are in the same Plane: So also are all the Parts of a Triangle.

Fig. 11.

I Say, the two Right-Lines AB, CD, meeting in the Point E, and the Triangle AEC, whose two Sides AE,

AE, CE, are parts of the two preceding Lines AB, CD, are in the same Plane. Plate 1. Fig. 11.

DEMONSTRATION.

If thro' the Point F taken at discretion in the Side CE, you draw a Right-Line AFG, to the opposite Angle A, by *Prop.* 1. the two Parts AF, FG, are in the same Plane, and so also are the two AE, EB, and CF, EF, and because the three Points E, F, C, are in a Right-Line by *Constr.* the three Lines AB, AG, CG must necessarily touch one another, as also the three Planes in which they are, and so become one.

Thus you may find that the Line AF is in the same Plane as the Side AE of the Triangle AEC; and after the same manner you may find that all the Right-Lines that can be drawn from the Angle A, thro' what other Points you please in the Side CE, are in the same Plane as they in the Side AE of the Triangle AEC. Whence 'tis easy to conclude that the Triangle AEC, as well as the two Lines AB, CD, are in the same Plane. *Which was to be demonstrated.*

USE.

This Proposition serves to demonstrate *Prop.* 4. and 5. that suppose two Right-Lines making an Angle to be in the same Plane. 'Tis of use in Perspective, to demonstrate that a Right-Line when projected upon a Plane, is a Right-Line, where we shall suppose, that all Right-Lines drawn from the Eye, thro' all the Points of a Right-Line, are in the same Plane, that is Triangular.

PROPOSITION III.

THEOREM III.

The common Section of two Planes is a Right-Line.

Plate 1. Fig. 3. 'TIS evident that the common Section of the two Planes ABCD, EFGH, is a Right-Line, because if thro' any two Points E, H, of this common Section, you draw in each Plane two Right-Lines, they will fall upon one another, because they can't bound a Space, and so they will make one Right-Line EH, which being common to the two Planes ABCD, EFGH, must be their common Section. *Which was to be demonstrated.*

USE.

This Proposition serves to demonstrate *Prop.* 4. 16. 18. and 19. that suppose the common Section of two Planes is a Right-Line. We shall also use it in Perspective, to demonstrate that a Right-Line projected on a Plane will be a Right-Line; and in Dialling, to demonstrate that all great Circles of a Sphere projected on a Plane, will be Right-Lines: It may be used also in other Projections, as to demonstrate that an intire great Circle, perpendicular to the Plane of Projection, when projected becomes a Right-Line.

PROPOSITION. IV.

THEOREM IV.

A Right-Line perpendicular to two others that intersect one another, will be the same to the Plane of those two Lines.

Fig. 4. I Say, if the Line AB be perpendicular to each of the two Right-Lines GH, IK, that are in the Plane CDEF, and intersect in the Point B, it will also be perpendicular to the Plane CDEF, that is to say, by *Def.* 3

to all the Lines drawn on the Plane thro' the Point B, to the Line LBM. Plate 1. Fig. 2.

PREPARATION.

Cut the equal Lines BG, BH, BI, BK, at discretion, and join the Right-Lines GI, KH. And draw from the Point A, thro' the Points I, L, G, K, M, H, as many Right-Lines.

DEMONSTRATION.

Because the four right-angled Triangles ABG, ABH, ABI, ABK, are equal, by 4. 1. the Bases AG, AH, AI, AK, will be equal; and for the same Reason the Isosceles Triangles GBI, KBH, being equal, their Bases GI, KH, will be equal, together with their Angles. Consequently by 26. 1. the equiangular Triangles LBG, MBH, will also be equal, and consequently the Side BL, is equal to the Side BM, and the Side GL to the Side HM, and by 8. 1. the Triangles AGI, AKH, are equal, and consequently the Angle AGI is equal to the Angle AHM. Wherefore by 4. 1. the two Triangles AGL, AHM are equal, consequently the Base AL is equal to the Base AM. Whence 'tis easy to conclude by 8. 1. that the Triangles ABL, ABM, are equal, and consequently the Angle ABL is equal to the Angle ABM, so that the Line AB is perpendicular to the Line LM. *Which was to be demonstrated.*

USE.

This Proposition serves to demonstrate *Prop.* 5. 8. 9. 11. and 15. and in Spherics, that a Right-Line passing thro' the Poles of a Circle, is perpendicular to the Plane of that Circle. It furnishes us also with a Method of letting fall a Perpendicular to a Plane, from a Point given without the Plane, different from that in *Prop.* 11. For instance, if you would let fall a Perpendicular to the Plane CDEF, from the Point A, describe upon the Point A, with any aperture of your Compass you please,

Plate 1. Fig. 2. the Circumference of a Circle on that Plane, and having marked at Pleasure three Points on that Surface, as G, H, I, for finding the Center B, draw thro' the Center B to the Point given A, the Right-Line AB, and that shall be perpendicular to the Plane proposed CDEF, the three Right-Lines AG, AH, AI, being equal. By this you may know whether a Stile, as AB, be placed right on the Plane CDEF, by taking at pleasure from its Foot the three equal Distances BG, BH, BI, for if it be well fixed, the Point B will be equidistant from the three Points G, H, I.

PROPOSITION V.

THEOREM V.

If one Right-Line be perpendicular to three others, intersecting one another in the same Point, those three will be in the same Plane.

Fig. 2. I Say, if the Right-Line AB, be perpendicular to the three Lines BC, BD, BF, intersecting one another in the Point B, these three Lines, BC, BD, BF, are in the same Plane: So that if the Plane of the two Lines BA, BF, be BAK, and the Plane of BC, and BD be DGHI, the Line BF will be the common Section of those two Planes.

DEMONSTRATION.

If the Line BE be the common Section of the two Planes DGHI, BAK, then by *Def.* 3. the Line AB being perpendicular to BD and BC, by *Sup.* and consequently to their Plane DGHI, by *Prop.* 4. It is also perpendicular to the common Section BE, and so the Angle ABE is right, consequently equal to the Angle ABF, which is also right, because the Line AB is supposed also to be perpendicular to the Line BF. Whence 'tis easy to conclude that the two Lines BE, BF, agree together, and consequently the Line BF is the common Section of the two Planes DGHI, BAK, so that it is in the Plane of the two Lines BC, BD. *Which was to be demonstrated.*

USE.

USE.

This Proposition is a Lemma to the following one.

PROPOSITION VI.

THEOREM VI.

Right-Lines perpendicular to the same Plane, are parallel to one another.

I Say, if the two Right-Lines AB, CD, are each perpendicular to the Plane EFGH, they are parallel to each other. Plate 1. Fig. 12.

PREPARATION.

Join the Right-Line BD, to which having drawn the perpendicular DI equal to AB, in the Plane EFGH, join the Right-Lines BI, AI, AD.

DEMONSTRATION.

Because the Line AB is perpendicular to the Plane EFGH, by *Sup.* it will also be perpendicular to the Line BD, by *Def.* 3. So that the Angle ABD being right, will be equal to the Angle BDI, that is also right by *Constr.* and because the Line DI was made equal to the Line AB, by 4. 1. the two right-angled Triangles ABD, DBI, are equal, and the Base AD equal to the Base BI; and then by 8. 1. the two Triangles AID, AIB, are equal, and the Angle ADI equal to the Angle ABI, which being right, by *Def.* 3. because the Line AB is perpendicular to the Plane EFGH, the Angle ADI must be right, and so ID perpendicular to AD, and since it is also perpendicular to the Line BD, by *Constr.* and to the Line CD, by *Def.* 3. the Line CD being supposed perpen-

Plate 1. Fig. 12. perpendicular to the Plane EFGH, the three Lines DC, DA, DB, to which the Line ID is perpendicular, are in the ſame Plane, by *Prop.* 5. conſequently the two Perpendiculars AB, CD, are alſo in the ſame Plane, and by 29. 1. they are parallel to one another. *Which was to be demonſtrated.*

USE.

This Propoſition ſerves to demonſtrate *Prop.* 9. 13. and 14. and ſhow that two Parallel Lines, as AB, CD, are in the ſame Plane, and this ſerves to demonſtrate *Prob.* 7. and 8. that ſuppoſes two parallel Lines to be in the ſame Plane.

PROPOSITION VII.

THEOREM VII.

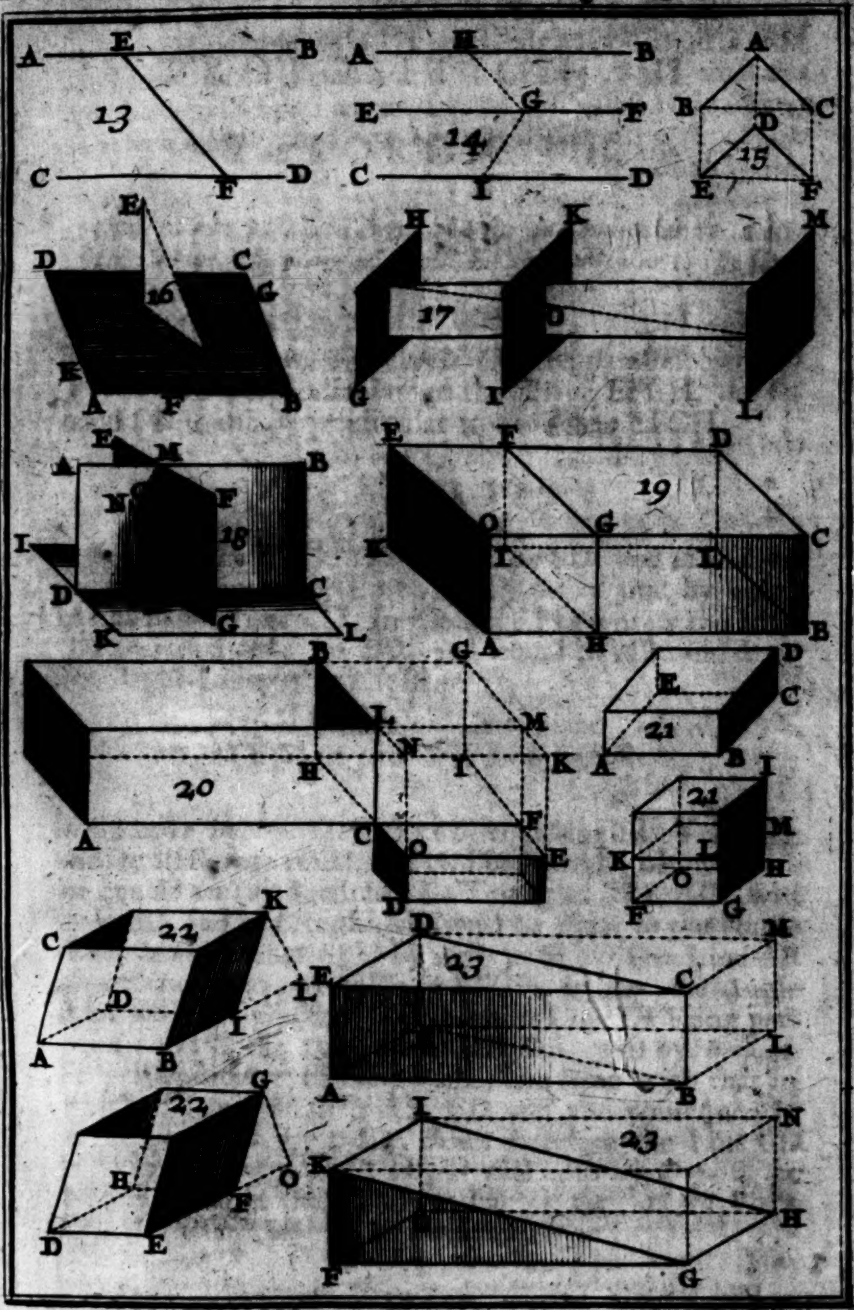
A Right-Line drawn from one parallel to another, is in the Plane of thoſe two Parallels.

Plate 2. Fig. 13. I Say, if thro' the Point E, of the Line AB, you draw to another Point F of the Line CD, parallel to the firſt AB, the Right-Line EF, that Right-Line EF, is in the Plane of theſe two parallel Lines AB, CD.

DEMONSTRATION.

Becauſe the two Points E, F, are in the Plane of the two Parallels AB, CD, a Right-Line may be drawn in this Plane thro' the Points E, F, that ſhall not differ from the Line EF, becauſe two Right-Lines can't bound a Space. So that the Line EF is in the Plane of the two Parallels AB, CD. *Which was to be demonſtrated.*

PRO-



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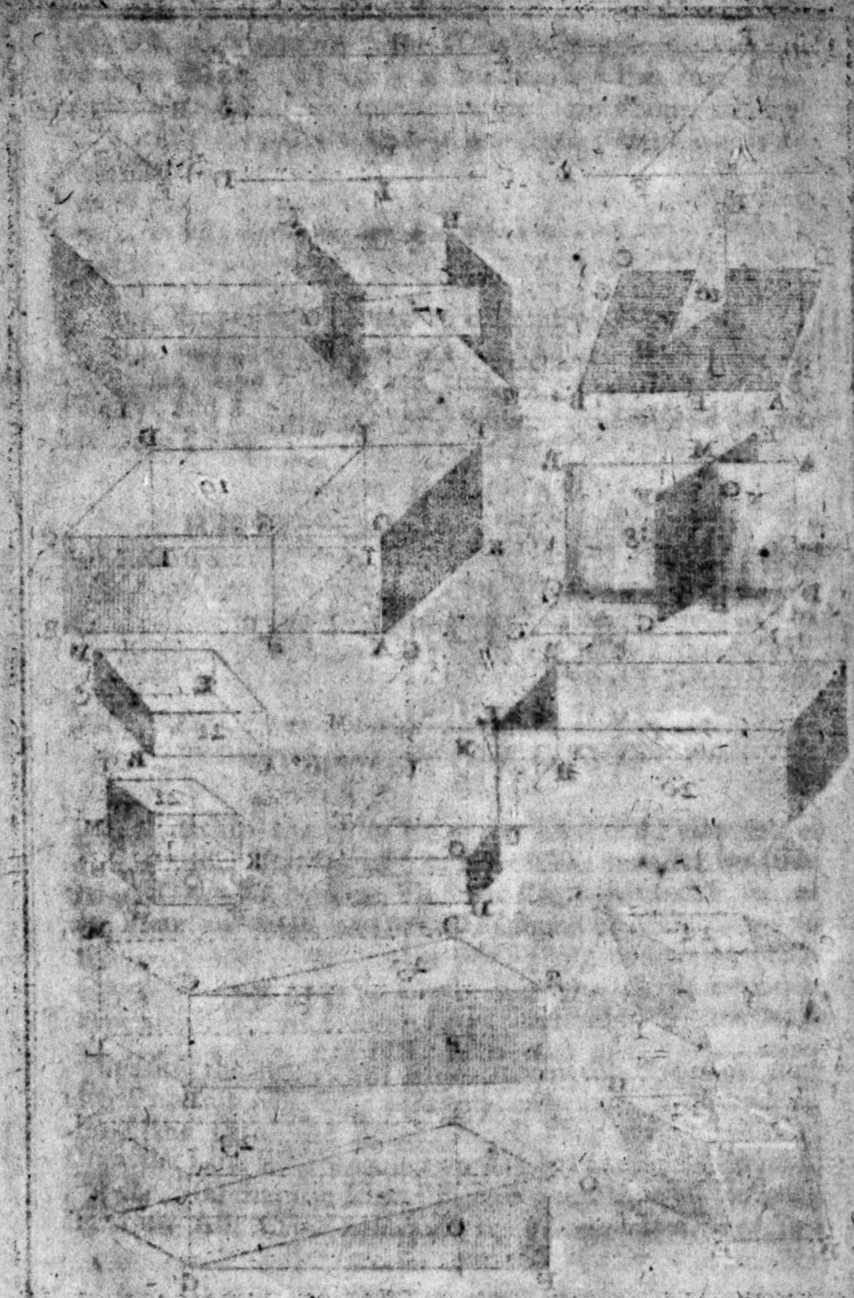

PROPOSITION VIII.

THEOREM VIII.

If there be two parallel Lines, the one perpendicular to a certain Plane, the other also will be perpendicular to the same Plane.

I Say, if the two Lines AB, CD, be parallel, and the first AB perpendicular to the Plane EFGH, the second CD is also perpendicular to the Plane EFGH. Plate 1. Fig. 12.

PREPARATION.

In the Plane EFGH draw the Line BD, and it will be perpendicular to the Line AB, by *Def.* 3. and by 29. 1. to the parallel one CD. In the same Plane draw the Line DI perpendicular to BD, and equal to AB, and draw the Right-Lines AD, AI, BI.

DEMONSTRATION.

Because by 4. 1. the two right-angled Triangles ABD, BDI, are equal, the two Bases AD, BI, will also be equal: and by 8. 1. the two Triangles ABI, ADI, will be equal, and the Angle ADI will be equal to the Angle ABI, which being right by *Def.* 3. Since the Line AB is perpendicular to the Plane EFGH, by *Sup.* the Angle ADI will be right also. So that the Line DI being perpendicular to the two Lines DB, DA, will by *Prop.* 4. be perpendicular to their Plane, the same with that in which the two parallels AB, CD are, and consequently to the Line CD, by *Def.* 3. Since therefore the Line CD is perpendicular to DB, DI, it will also by *Prop.* 4. be perpendicular to their Plane, that is say, to the Plane EFGH. *Which was to be demonstrated.*

USE.

This Proposition serves to demonstrate *Prop.* 9, 10, 11, 12, and 18.

PRO-

PROPOSITION IX.

THEOREM IX.

Two Right-Lines parallel to a third, are parallel to one another, tho' they be not in the same Plane.

Plate 2. Fig. 14. I Say, if the Lines AB, CD, be parallel each to the same Line EF, they are so to one another, tho' they be not in the same Plane, otherwise this Theorem would be evident by 30. 1.

PREPARATION.

Draw thro' the Point G, taken at discretion in the Line EF, in the Plane of the two Parallels AB, EF, the Line GH, perpendicular to the Line EF, and it will be perpendicular also to the Line AB, by 29. 1. and in the Plane of the two Parallels EF, CD, the Line GI perpendicular to the same Line EF, and it will be perpendicular to the Line CD, by 29. 1.

DEMONSTRATION.

Because the Line EG is perpendicular to each of the two Lines GH, GI, by *Constr.* it will be perpendicular to their Plane, by *Prop.* 4. consequently by *Prop.* 8. the the two Lines AB, CD, that are parallel to the Line EG, by *Sup.* will also be perpendicular to the same Plane of the two Lines GH, GI, and by *Prop.* 6. the two Lines AB, CD, will be parallel to one another. *Which was to be demonstrated.*

USE.

This Proposition serves to demonstrate the following, and *Prop.* 15. and is used in Dialling, to demonstrate that in different Dials, the Axes are parallel to one another, because they are so to the Axis of the World.

PRO-

PROPOSITION X.

THEOREM X.

If two Right-Lines, making an Angle, are parallel to two others of a different Plane, the two others will form an Angle equal to that of the two former.

I Say, if the two Lines A B, AC, are parallel to these two DE, DF, the Angle BAC is equal to the Angle EDF, tho' the Plane of the two Lines AB, AC, be different from that of the two Lines DE, DF. Plate 2. Fig. 15.

PREPARATION.

Cut off the Line DE equal to the Line AB, and the Line DF equal to the Line AC, and join the Right-Lines BC, EF, BE, AD, CF.

DEMONSTRATION.

Becauſe the two Lines AB, DE, are parallel by *Sup.* and equal by *Conſt.* the two Lines AD, BE, will alſo be equal and parallel, by 33. 1. and for the ſame reaſon AD, CF, will be equal and parallel: Conſequently BE, CF will be equal, by Ax. 1. and parallel by *Prop.* 9. and by 33. 1. BC, EF, will be equal. And laſtly, by 8. 1. the two Triangles ABC, DEF, will be equal, and the Angle BAC equal to the Angle EDF. *Which was to be demonſtrated.*

USE.

This Propoſition is uſed in Perſpective to demonſtrate that two Right-Lines parallel to the Plane of Projection, when projected, form an Angle equal to that of the two Right-Lines; and that two Right-Lines when projected, are parallel to one another, if the two Right-Lines are parallel to one another and the Plane of Projection, *Prop.* 24. is demonſtrated alſo by the help of this.

PROPOSITION XI.

PROBLEM I.

To let fall a Right-Line from a Point given without a Plane, perpendicular to it.

Plate 2. Fig. 16.

TO let fall a Perpendicular to the Plane ABCD, from the Point E, given without the Plane: draw at discretion in the Plane, the Right-Line FG, and let fall perpendicular to it, the Line EH from the Point E, by 12. 1. draw also from the Point H, the Right-Line HI perpendicular to the Line FG, by 11. 1. and by 12. 1. the Perpendicular EI, to the Line HI, from the Point given E, and it will be perpendicular to the Plane proposed.

DEMONSTRATION.

Because the Line FG is perpendicular to HI and HE, by *Constr.* it will be so also to their Plane EHI, by *Prop.* 4. Consequently, draw IK parallel to the Line FG, and you will find by *Prop.* 8. that it is perpendicular also to the Plane EHI, and consequently to the Line EI, by *Def.* 3. Since therefore the Line EI is perpendicular to IK and IH, it is perpendicular also by *Prop.* 4. to their Plane ABCD. *Which was to be demonstrated.*

USE.

This Proposition serves as a Lemma to the following one; and I shall use it pretty often in Dialling, when in drawing a Dyal upon a Wall, having determined the extremity of the Stile at the Point of a Wire planted obliquely on the Wall, I would determine its Foot and Length.

PRO-

PROPOSITION XII.

PROBLEM II.

To erect a Line perpendicular to a Plane from a Point given in the Plane.

TO erect a Line from the Point B, in the Plane EFGH, perpendicular to that Plane; let fall by *Prop.* 11. from the Point C, taken at discretion without the Plane, the Perpendicular CD, and thro' the Point B, draw by 30. 1. the Line AB parallel to the Line CD, and it will be perpendicular to the Plane proposed EFGH, as is evident by *Prop.* 8. Plate 1. Fig. 12.

USE.

This Proposition serves in Dialling for placing the Stile in a Dial described on a Plane: But 'tis better to use a Square, drawing from the Foot of the Stile B, two Lines at discretion BD, BI, in the Plane of the Dial EFGH, to apply to it the Side of the Square, so that the Right-Angle touch the Point B, and place the Stile AB, so that it touch the other Side of the Square, for by that means it will be perpendicular to the two Lines BD, BI, and consequently to their Plane EFGH, by *Prop.* 4.

PROPOSITION XIII.

THEOREM XI.

Two Right-Lines can't be drawn perpendicular to a Plane, thro' the same Point.

I Say, first, that from the Point D, taken in the Plane EFGH, two different Right-Lines can't be drawn perpendicular to this Plane, for instance DC, DA; because these two Lines would be parallel to each other, by *Prop.* 6. and so would coincide, and form but one and the same Line, since they proceed from the same Point D.

I

Plate 1. Fig. 12. I say, in the second Place, that from the Point A, taken without the Plane EFGH, two different Right-Lines can't be drawn perpendicular to this same Plane, for instance AB, AD, as well on the account of what has been said, as because these two Perpendiculars AB, AD, being in the same Plane, by *Prop.* 3. whose Section with the Plane EFGH, will be BD, they will make with that common Section BD, two Right-Angles by *Def.* 3. so that each of these two Angles ABD, ADB, of the Triangle DAB, would be right, which is impossible, by 32. 1.

USE.

This Proposition is so evident, that it deserves not to be mentioned, and *Euclid* seems unwilling to have added it, were it not to demonstrate by the help of it, *Prop.* 19. and 38.

PROPOSITION XIV.

THEOREM XII.

Those Planes are parallel, that have the same Right-Line perpendicular to them.

Fig. 5. I Say, if the Line IK be perpendicular to each of the two Planes, ABCD, EFGH, these two Planes are parallel, that is to say, equidistant by *Def.* 8. So that if you draw the Line DI parallel to the Line IK, it being perpendicular at the same time to the two Planes ABCD, EFGH, by *Prop.* 6. the two Parallel Lines IK, DL, will be equal.

DEMONSTRATION.

Join the Right-Lines, ID, KL, and you will find by *Def.* 3. that the four Angles of a Figure DIKL are right, and consequently is a Parallelogram, wherefore by 34. 1. the two opposite Sides IK, DL, will be equal. *Which was to be demonstrated.*

USE.

This Proposition shews us that all the Circles of a Sphere, having the same Poles, are parallel, because they have

have the same Axis, perpendicular to them: We shall make use of this Proposition in the Demonstration of the following one.

PROPOSITION XV.

THEOREM XIII.

If the two Legs of one Angle are parallel to the two Legs of another in a different Plane, the Planes of these two Angles will be parallel.

I Say, if the Lines IM, IN, of the Angle MIN, in the Plane ABCD, are Parallel to the two Lines GP, GE, of the Angle PGE, in the Plane EFGH, the two Planes ABCD, EFGH are Parallel. Plate 5. Fig. 5.

PREPARATION.

Let fall the Line IK perpendicular to the Plane EFGH, from the Point I, by *Prop.* 11. and thro' the Point K, where it meets the Plane, draw in the same Plane the two Lines KO, KQ, parallel to GP, GE, and by consequence to IM, IN, by *Prop.* 9.

DEMONSTRATION.

Because the Line IK is perpendicular to the Plane EFGH, by *Constr.* each of the two Angles IKO, IKQ will be right, by *Def.* 3. and because the two Lines KO, IM are parallel by *Constr.* and consequently in the same Plane, by *Prop.* 6. the Angle KIM will be also right, by 29. 1. After the same manner you may find the Angle KIN is right, because KQ, IN are parallel. Wherefore the Line IK, being perpendicular to IM, and IN, will also be perpendicular to their Plane ABCD, by *Prop.* 4. and

Plate 1. because 'tis perpendicular also to the Plane EFGH, by Constr. it follows by *Prop.* 14. that the two Planes ABCD, EFGH, are parallel. *Which was to be demonstrated.*

PROPOSITION XVI.

THEOREM XIV.

The common Sections of one Plane, with two other parallel Planes, are also parallel.

Fig. 5. 'TIS plain the two common Sections ID, KL, of the Plane DIKL, with the two parallel Planes ABCD, EFGH, are parallel, because being in the parallel Planes ABCD, EFGH, they cannot get out of it, by *Prop.* 1. and so can never meet.

USE.

This Proposition serves to demonstrate the following, and *Prop.* 16. and 24. and in Perspective, to demonstrate that Lines parallel to a Plane of Projection, are so also when projected.

PROPOSITION XVII.

THEOREM XV.

Two Right-Lines are cut proportionally by parallel Planes.

Plate 2. Fig 17. I Say, the two Right-Lines AB, CD, are divided proportionally by the Parallel Planes GH, IK, LM, that is to say, the Ratio of the Parts AE, EB, is equal to that of CF, FD.

DEMONSTRATION.

Draw the Right-Line AD, meeting the Plane IK in the Point O, and by *Prop.* 16. you will find the common Sections EO, BD, of the Triangular Plane ABD, with the two parallel Planes IK, LM, to be Parallel, and by 2. 6.

2. 6. the Ratio of the two Lines AO, OD, equal to the Ratio of the two Lines AO, OD. In like manner, you may find that the common Sections AC, OF, of the Triangular Plane ADC, with the two parallel Planes GH, IK are parallel, and consequently the Ratio of the two Lines CF, FD, is equal to that of the two Lines AO, OD; that is to say, to the two AE, FD. *Which was to be demonstrated.* Plate 2. Fig. 17.

PROPOSITION XVIII.

THEOREM XVI.

If a Right-Line be perpendicular to a Plane, all the Planes it can be found in, are also perpendicular to that Plane.

Plate I. Fig. 3.

I Say, if the Line IK be perpendicular to the Plane ABCD, any Plane whatever wherein 'tis found, for instance the Plane EFGH, whose common Section with the Plane ABCD, is the Right-Line EH, will be perpendicular to the Plane ABCD.

DEMONSTRATION.

Draw in the Plane EFGH, any Line as GH, perpendicular to the common Section EH, by 29. 1. you will find it parallel to the Line IK, which being perpendicular to the Plane ABCD, by *Sup.* makes it evident by *Prop.* 8. that the Parallel GH, is also perpendicular to the Plane ABCD, and by *Def.* 4. that the Plane EFGH is perpendicular to the Plane ABCD. *Which was to be demonstrated.*

USE.

This Proposition serves to demonstrate that all the great Circles of a Sphere, passing thro' the Poles of another, are perpendicular to the Poles of that other; and that all vertical Circles are perpendicular to the Plane of the Horizon. Lastly, That all Meridional Circles are perpendicular to the Plane of the Equator.

PROPOSITION XIX.

THEOREM XVII.

If two intersecting Planes, be perpendicular to another, their common Section also will be perpendicular.

Plate 2: Fig. 18. I Say, if each of these two Planes ABCD, EFGH, whose common Section is MH, be perpendicular to the Plane IKLC, their common Section MH, will also be perpendicular to that Plane.

PREPARATION.

Draw from the Point H, in the Plane ABCD, the Right-Line HN, perpendicular to the common Section DH of this Plane, with the Plane IKLC, and in the Plane EFGH, the Right-Line HO, perpendicular to the common Section GH, of that Plane, with the Plane IKLC.

DEMONSTRATION.

Because the two Lines HN, HO, are by *Constr.* perpendicular to the common Sections DH, GH, of the Plane IKLC, with the Planes ABCD, EFGH, that are perpendicular to the Plane IKLC, by *Sup.* they would be perpendicular by *Def.* 4. to the same Plane IKLC, but that being impossible by *Prop.* 13. these two Perpendiculars HN, HO, must become one, namely HM, which by consequence is perpendicular to the Plane IKLC. *Which was to be demonstrated.*

USE.

This Proposition is of use in Perspective, to demonstrate, that when the Plane of Projection is right, that is to say, is perpendicular to the Geometric Plane, Right-Lines perpendicular to the Geometric Plane, when projected, become Right-Lines perpendicular to the Ground.

PRO-

PROPOSITION XX.

THEOREM XVIII.

If three Plane Angles form a ſolid one, the Sum of any two is greater than the third.

I Say, if the three Plane Angles BAC, BAD, CAD, form the ſolid Angle A, the greateſt for inſtance BAC, is leſs than the Sum of the two others BAD, CAD. Plate 1. Fig. 6.

CONSTRUCTION.

Cut off from the greateſt Angle BAC, the Angle BAE, equal to the Angle BAD, and making the Lines AD, AE equal, join the Right-Lines, BEC, DB, DC.

DEMONSTRATION.

Becauſe the Angle BAE is equal to the Angle BAD, by *Conſtr.* and the Side AE equal to the Side AD, the Triangles BAD, BAE, will be equal by 4. 1. and the Baſe BE, equal to the Baſe BD; and ſince the Sides DB, DC, of the Triangle BDC, taken together, are greater than the ſingle Side BC, by 20. 1. taking away the equal Lines BD, BE, there will remain the Line CD, greater than the Line CE, and by 25. 1. the Angle CAD will be greater than the Angle CAE. Wherefore adding the two equal Angles BAD, BAE, you will find the two Angles CAD, BAD are taken together greater than the Angle BAC. *Which was to be demonſtrated.*

USE.

This Propoſition ſerves to demonſtrate the following, though that may be demonſtrated without it, as you ſhall ſee.

PROPOSITION XXI.

THEOREM XIX.

All the Plane Angles that form a solid one, taken together, are less than four right.

Plate 1. Fig. 6. I Say, the Sum of the three plane Angles BAC, BAD, CAD, that form the solid Angle A, are together less than four right.

DEMONSTRATION.

If the three Plane Angles BAC, BAD, CAD, were in the Plane BCD, they would be together equal to four right, because measur'd by the Circumference of a Circle described upon their common Point A; but since the Angles are raised above the Plane BCD, and consequently less than if they were upon that Plane, as 'tis plain from 21. 1. the three Angles BAC, BAD, CAD, together, must be less than four right. *Which was to be demonstrated.*

The XXII *and* XXIII *Propositions are needless.*

PROPOSITION XXIV.

THEOREM XXI.

If a Solid be bounded by parallel Planes on four Sides, the opposite ones will be similar and equal Parallelograms.

Fig. 1. I Say, if the solid ABCDE, be bounded by parallel Planes, on four Sides, its opposite Surfaces are similar and equal Parallelograms.

DEMONSTRATION.

Because the Planes AEGF, BCDH, are parallel by *Constr.* and cut by the Plane DEFH, the common Sections EF, DH, will be parallel by *Prop.* 16. and so because

cause the Planes ABHF, CDEG, are parallel, and cut by the Plane DEFH, the common Sections ED, FH, will be parallel. Which shows that the Plane DEFH is a Parallelogram; and thus also you may find, that the other Planes are Parallelograms: Whence one may easily conclude, that the two opposite ones are equiangular, by *Prop.* 10. and equal, because they have equal Sides 16. 9. 34. 1. *Which was to be demonstrated.* Plate 1. Fig. 1.

USE.

This Proposition serves as a Lemma to the next, and to demonstrate *Prop.* 28.

PROPOSITION XXV.

THEOREM XXII.

If a Parallelopiped be cut by a Plane parallel to one of its Surfaces; the two Solids that are form'd by that Division, will be to one another as their Bases.

I Say, if you divide the Parallelopiped ABCDE, by the Plane FGHI, parallel to the Plane AOEK, or BCDL, the Solid EFGHA, will be to the Solid FDCBH, as the Base AHIK, to the Base HILB. Plate 2. Fig. 19.

DEMONSTRATION.

Imagine Planes parallel to the common Base ABLK, or CDEO, to pass thro' all the Points of the Line AO, that may be taken for the common Height of the two Solids EH, FB, that are Parallelopipeds, by *Prop.* 24. and these Planes will divide each Solid into an equal Number of little Planes, that are Parallelograms equal and similar to the Base of its Parallelopiped, by *Pro.* 24. So that each Plane of the solid EH, will have the same Ratio to each Plane of the solid FB, as the Base AI, has to the Base HL, and by 12. 5. all the Planes of the Solid EH, that is to say, the Solid EH will have the

 same

Plate 2. Fig. 19. same Ratio to all the Planes of the Solid FB, that is to say, to the Solid FB, as the Base AI has to the Base HI. *Which was to be demonstrated.*

USE.

This Proposition shows us that Parallelopipeds of the same Height, are to one another as their Bases; which ought to be extended to Prisms too, because the Demonstration will serve there, if the two opposite Planes that are parallel, similar and equal, be consider'd as Bases.

Proposition XXVI. *and* XXVII. *are needless.*

PROPOSITION XXVIII.

THEOREM XXIII.

A Parallelopiped is divided into two equal Prisms, by a Plane that passes thro' the two Diagonals of the two opposite Surfaces.

Plate 1. Fig. 1. I Say, the Parallelopiped ABCDE, is divided into two equal Parts by a Plane passing thro' the two parallel Diagonals AC, FD, of the two opposite Surfaces, ABCG, DEFH.

DEMONSTRATION.

Imagine Planes parallel to the Base ABCG, passing thro' all the Points of the Line AF, that may be looked upon as the Height of the Parallelopiped ABE, and they will divide the Parallelopiped ABE, into little Parallelograms similar and equal to the Base ABCG, by *Prop.* 24. and by 34. 1. they will be divided each into two equal Triangles by the Plane that passes thro' the two Diagonals AC, FD. Which shows that the two Triangular Prisms arising from the Section of the Parallelopiped ABCDE, by the Diagonal Plane, contains an equal Number of Triangles, and consequently are equal. *Which was to be demonstrated.*

USE.

USE.

This Proposition serves to demonstrate *Prop.* XL.

Prop. XXIX. *is needless, because virtually contain'd in the two next, that we have reduced into one.*

PROPOSITION. XXX. and XXXI.

THEOREM XXV. and XXVI.

Parallelopipeds of the same Height, having the same Base, or equal Bases, are equal.

IT naturally follows from *Prop.* 25. where we found that Parallelopipeds of the same Height are to one another as their Bases; from whence 'tis easy to conclude that when the Bases are equal, the Parallelopipeds are equal. 'Tis the same in Prisms.

PROPOSITION XXXII.

THEOREM XXVII.

Parallelopipeds of the same Height, are as their Bases.

THis also follows from *Prop.* 25. that shows this Theorem is also true of Prisms.

PRO-

PROPOSITION XXXIII.

THEOREM XXVIII.

Similar Parallelopipeds are in the triplicate Ratio of their Homologous Sides.

Plate 2. Fig. 20.

I Say, if the Parallelopipeds ABLC, CDEF are ſimilar, all the Planes of the one being ſimilar to all the Planes of the other, and all their Angles equal. In which Caſe the Solids may be plac'd in a Right-Line, as may be ſeen in the Figure, theſe Parallelopipeds will be in the triplicate Ratio of that of their Homologous Sides, for inſtance, AC, CF.

DEMONSTRATION.

Deſcribe the Parallelopipeds CG, OM, by producing the Sides of the two propoſed, as you ſee in the Figure, then by *Prop.* 32. the ſolid ABLC, is to the Solid BCFG, of the ſame Height, as the Baſe AH, to the Baſe CI, or by 6. 1. as the Side AC, to the Side CF: And thus you may find, that the Solid BCFG, is to the Solid CEKL, as the Baſe CI is to the Baſe CE, or as the Side CH is to the Side CO. And laſtly, That the Solid CEKL is to the Solid CDEF, as the Baſe OK to the Baſe DE, or as the Side ON is to the Side OD; but ſince the Ratio of ON to OD is the ſame as that of CH to CO, and that of AC to CF, by *Sup.* It follows that the Ratio of the Solid ABLC to the Solid CDEF being compounded of three equal Ratios, muſt be the triplicate of each, and conſequently of that of AC to CF. *Which was to be demonſtrated.*

COROLLARY. I.

It follows from this Propoſition, that ſimilar Parallelopipeds are as the Cubes of their Homologous Sides, becauſe the Cubes being ſimilar Parallelopipeds are in the

the Triplicate Ratio of that of their Homologous Sides.

COROLLARY II.

From hence alſo it follows, that if four Lines be in continual Propoſition, a Parallelopiped deſcribed on the firſt, is to a ſimilar one deſcribed on the ſecond, as the firſt Line is to the fourth, becauſe the Ratio of the firſt to the fourth is the triplicate of that of the firſt to the ſecond.

COROLLARY III.

Laſtly, Similar Triangular Priſms are in the Triplicate Ratio of that of their Homologous Sides, becauſe by *Prop.* 28. they are halves of ſimilar Parallelopipeds, that are in this Triplicate Ratio. 'Tis the ſame alſo in ſimilar Polygonal Priſms, becauſe they may be reduced into Triangular Priſms.

USE.

This Propoſition ſerves to augment or diminiſh a Solid; for inſtance a Cube, according to a given Ratio. As if you would have a Cube double another propoſed, which is commonly call'd the Duplication of the Cube; find two continual mean proportional between the Side of the Cube propoſed and its double, and then the next Proportional will be the Side of the Cube, that is double the propoſed one, as is evident by *Corol.* 2. This Propoſition is uſed in demonſtrating *Prop.* 37.

By this Propoſition alſo you find, that if a Cube weigh a Pound for inſtance, a Cube of homogeneous Matter, whoſe Side is double that of the former, will weigh eight Pounds, becauſe the Triplicate of the double is the Octuple. And thus alſo a Sphere, whoſe Diameter is double that of another, will be eight times greater, becauſe two Spheres are in the triplicate Ratio of that of their Diameters, by 18. 12. This Propoſition is uſed in demonſtrating *Prop.* 8. 12. and 12. 12.

PRO-

PROPOSITION XXXIV.

THEOREM XXIX.

Equal Parallelopipeds have their Bases and Heights reciprocal; and such as have their Bases and Heights reciprocal, are equal.

Plate 2. Fig. 21.

I Say, first, if the Parallelopipeds ABCD, FGHI, be equal, their Bases and Heights are reciprocal, that is to say, the Base ABCE, is to the Base FGHO, as the Height HI, to the Height CD.

PREPARATION.

Taking HM equal to CD, make the Plane MLK, pass thro' the Point M, parallel to the Base FGHO.

DEMONSTRATION.

Because the Solid AD, is to the Solid FM of the same Height by *Constr.* as the Base AC is to the Base FH, by *Prop.* 32. the Solid FI is equal to the Solid AD, by *Sup.* is also to the Solid FM, as the Base AC, to the Base FM, by 7. 5. and because by *Prop.* 32. the Solid FI is to the Solid FM, as the Base GI to the Base GM, or by 1. 6. as the Height HI, to the Height HM or CD, its equal, by *Constr.* it follows by 11. 5. that the Base AC is to the Base FH, as the Height HI, to the Height CD. *Which was to be demonstrated.*

I say, in the second Place, if the Base AC be to the Base FH, as the Height HI is to the Height CD, the two Parallelopipeds AD, FI, are equal.

DEMONSTRATION.

Because the Base AC is to the Base FH, as the Height HI to the Height CD, or HM by *Sup.* and by *Prop.* 32. the Base AC is to the Base FH, as the Solid AD, to the Solid FM of the same Height; the Solid AD will be to the Solid FM, as the Height HI to the Height HM,

and

and because the Height HI is to the Height HM, as the Base GI is to the Base GM, by 1. 6. or, as the Solid FI to the Solid FM, by 32. the Solid AD must be to the Solid FM, as the Solid FI is to the Solid FM. and by 9. 5. the Solids AD, FI, are equal. *Which remain'd to be demonstrated.* Plate 2. Fig. 21.

SCHOLIUM.

These two Demonstrations suppose that the Parallelopipeds proposed AD, FI, are right-angled, so that the Sides CD, HI, may be taken for their Heights, but when that does not happen, that is to say, when the Sides CD, HI, are not perpendiculur to their Bases AC, FH, still the Demonstration will be the same, because by *Prop.* 28. you may imagine right-angled Parallelopipeds equal to the proposed ones upon the same Bases, by making them of the same Height. 'Tis plain also, this Theorem may be applied to all Sorts of Prisms, without enlarging upon it.

USE.

This Proposition serves to change a given Prism into another, on a given Base; thus if you would make a Prism on the Base ABCE, equal to the given Prism FI, find the Line CD a fourth proportional to the Base AC, the Base FH, and the Height HI, and that shall be the Height of the Prism sought, *&c.* It is used also to make out the 9. 12.

The XXXV *Prop. is needless.*

PROPOSITION XXXVI.

THEOREM XXXI.

If three Right-Lines be proportional, the Parallelopiped of these three Right-Lines, is equal to a Parallelopiped that is equiangular, and has all its Sides equal to the middle Line.

I Say, if the Lines AB, AC, AD, are proportional, the Parallelopiped ABKC, made by those three Lines, that is to say, whose three Dimensions are equal to them, Fig. 22.

is

Plate 2. Fig. 22. is equal to the equiangular Parallelopiped DEFGH, each of whose Sides is equal to the mean Proportional AC.

DEMONSTRATION.

Because each of the two Sides DE, EF, is equal to the Line AC, and the three Lines AB, AC, AD, proportionals, by *Sup.* AB is to DE, as EF to AD, and by 14. 6. the two Bases ABID, DEFG, supposed to be equiangular, are equal: and because the Heights KL, GO, are equal, the Angles F, I, being equal; and the Sides FG, IK, equal by *Sup.* Then by *Prop.* 31. the Solids AK, DG are equal. *Which was to be demonstrated.*

USE.

This Proposition is very useful in Arithmetic, to find the Side of a Cube equal to the Sum or Difference of two given Cubes, tho' indeed it may be done otherwise, without this Proposition.

PROPOSITION XXXVII.

THEOREM XXXII.

Similar Parallelopipeds described on Proportional Lines, are proportional, and if the similar Parallelopipeds be proportional, the Homologous Sides will also be proportional.

THe Demonstration of this Proposition, is entirely the same with that of similar Polygons in 22. 6. only using the triplicate Ratio instead of the duplicate, because similar Parallelopipeds are in the triplicate Ratio of that of their Homologous Sides, by *Prop.* 33. 'tis needless therefore to insist any longer on it.

PRO-

PROPOSITION XXXVIII.

THEOREM XXXIII.

If two Planes be perpendicular to one another, a Perpendicular let fall from a Point in one of these Planes to the other, will fall upon the common Section of the Planes.

I Say, if you let fall from the Point I, taken in the Plane EFGH, the Line IK, perpendicular to the Plane ABCD, which is suppos'd perpendicular to the Plane EFGH, the Point I is in the Perpendicular IK, will fall upon the common Section EH. Plate 1. Fig. 3.

DEMONSTRATION

A Perpendicular let fall from the Point I, in the Plane EFGH, to the common Section EH, will be perpendicular to the Plane ABCD, by *Def.* 4. and because by *Prop.* 13. two Perpendiculars can't be drawn to the same Plane, that same perpendicular will coincide with the first IK, and so will meet the common Section EH. *Which was to be demonstrated.*

USE.

This is very useful in the Orthographic Projection of a Sphere, *to demonstrate that a Circle perpendicular to the Plane of Projection, is represented by a Right-Line*; and in Dialling, *that a great Circle perpendicular to the Plane of the Dial, is represented by a Right-Line passing thro' the Foot of the Style.*

This Proposition seems to be misplac'd, for it respects only Lines and Planes, and ought to be plac'd at the beginning of the Book, at least after *Prop.* 13. that serves to demonstrate it.

I omit Prop. XXXIX. *because of no great Consequence.*

PROPOSITION XL.

THEOREM XXXV.

A Priſm, whoſe Baſe is a Parallelogram double the Triangular Baſe of another Priſm of the ſame Height, is equal to that other Triangular Priſm.

Plate 2. Fig. 23.

I Say, if the Heights AE, FK, of the two Triangular Priſms ABCDE, FGHIK, are equal, and the Baſe FGHO of the ſecond, be a Parallelogram double the Triangular Baſe ABP of the firſt, theſe two Priſms are equal.

DEMONSTRATION.

Compleat the Parallelogram ABLP, and it will be double the Parallelogram ABP, by 34. 1. and conſequently equal to the Parallelogram FGHO, that is alſo double the Triangle ABP, by *Sup.* Then compleat the Parallelopipeds ABMD, FGNI, and you will find by *Prop.* 31. the two Parallelopipeds are equal, and conſequently the Priſms ABD, FGI, their halves, by *Prop.* 28. are alſo equal. *Which was to be demonſtrated.*

USE.

This Propoſition ſhews how to find the Solidity of a Triangular Priſm, by multiplying its Triangular Baſe by its Height, or if you take one of its other Surfaces that are Parallelograms, for a Baſe, by multiplying that Baſe by half the Height, becauſe multiplying by the whole Height, you find the Solidity of a Parallelopiped, that is double the Priſm. Upon this Principle Sloaping Bodies are meaſured, as you will find in the *Practical Geometry*.

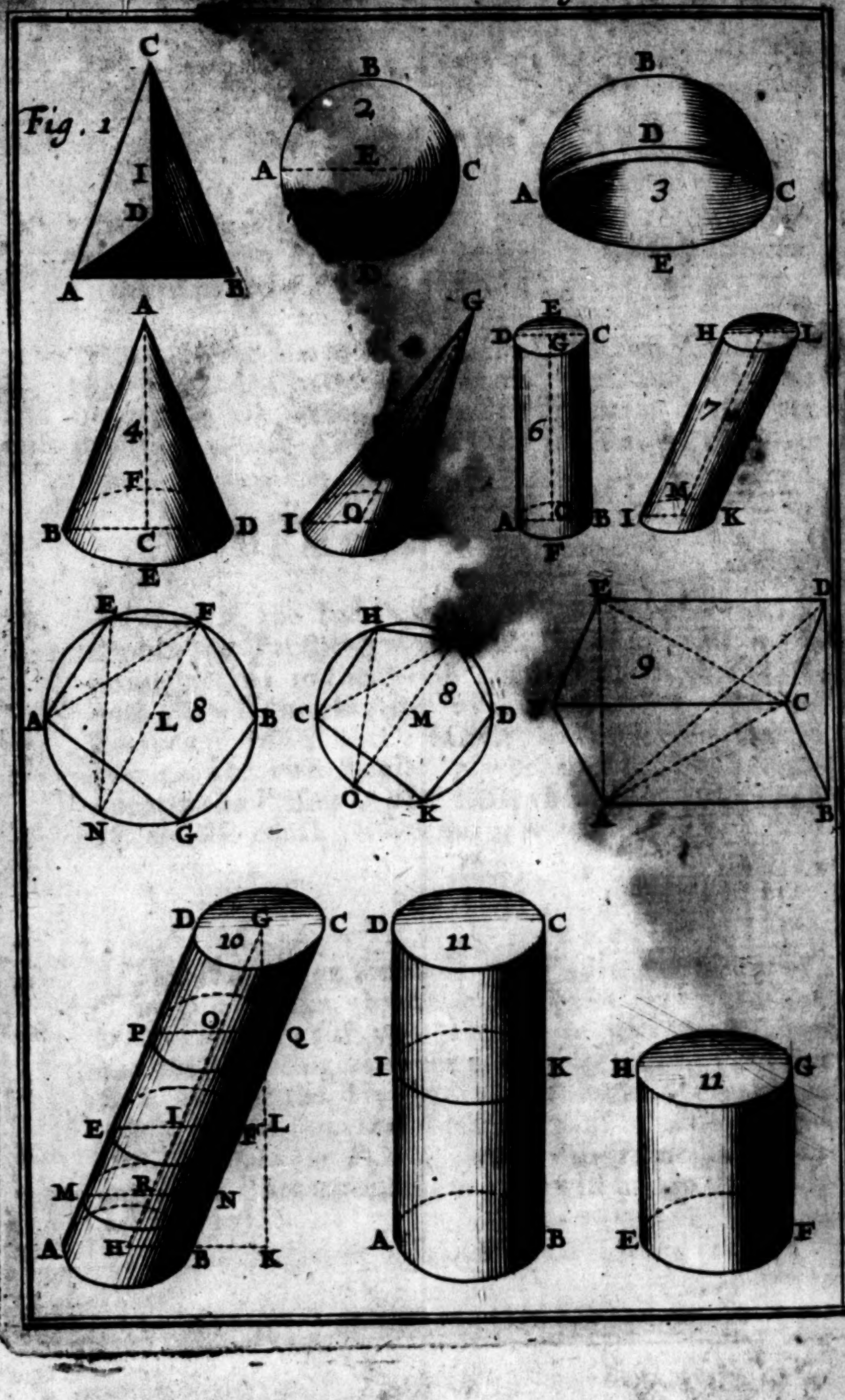
Fig. 1

THE TWELFTH BOOK OF EUCLID's ELEMENTS.

EUclid having treated of Prisms, and Parallelopipeds in the former Book, explains in this the Properties of other Bodies that are more difficult, namely such as are bounded by Curve Surfaces, as the Cone, Cylinder, and Sphere, concerning which the great *Archimedes* has given us very neat Demonstrations.

DEFINITIONS.

I.

A Pyramid is a Body bounded by several Triangular Planes meeting in the same Point, and having another Plane for the Base: *As* ABCD, *call'd a* Triangular Prism, *because its Base ABC is a Triangle, a Pyramid taking its Name from the Figure of the Base.* Fig. 1.

'Tis evident a Pyramid must have four Surfaces at least, including the Base, from whence the Pyramid is call'd a *Tetraedrum*, if its Triangles are equal and equilateral

II.

A Sphere is a Solid bounded by one Surface, having a certain Point in it, from whence all Right-Lines drawn to the Surface are equal: as ABCD.

'Tis plain a Sphere is generated by the intire Revolution of a Semicircle upon its Diameter. Thus imagine Fig. 2.

 the

Fig. 2. the Semicircle ABC, to move round the Diameter AC, till its Circumference ABC come to the Place where it began to move, and then its Motion will generate the Sphere ABCD.

III.

The Axe of a Sphere is that Right-Line or immoveable Diameter that the Semicircle is ſuppos'd to revolve about, in generating the Sphere: as AC.

This Line is call'd ſo from the Latin Word *Axis*, that ſignifies an Axle-Tree.

IV.

The Center of a Sphere is that Point from which all Right-Lines drawn to the Surface, are equal: as E.

Fig. 3. 'Tis evident that if a Sphere be cut by a Plane paſſing thro' its Center, the Section will be a Circle, as ADCE, and the Sphere will be divided into two equal Parts, call'd *Hemiſpheres*, as ABCD, whoſe external Surface is call'd the *Convex Surface*, and the internal Surface, its *Concave Surface*.

V.

The Diameter of a Sphere, is a Right-Line drawn thro' the Center of the Sphere, and bounded on each Side by its Surface: as AC.

Fig. 2. 'Tis evident that every Axe is a Diameter, but not every Diameter an Axe. 'Tis evident alſo that a Sphere as well as a Circle, has an infinite Number of Diameters, all equal to one another, whoſe Halves iſſuing from the Center, and terminated by the Surface, are call'd, *Semi-diameters*, or *Radii*, as in a Circle.

VI.

A *Cone* is a Solid bounded by two Superficies, produced by the intire Revolution of a right-angled Triangle, about one of its Sides, forming the Right-Angle.

Thus

Thus suppose the Right-Angled Triangle ACD revolve round the immoveable Side AC, so that the Circumvolution be perfect; that is to say, the Side CD, stop at the Place it began to move in, and the Triangle ACD will describe by that intire Revolution the Cone ABED, call'd a Right-angled Cone; *if the right-angled Triangle ACD, call'd the* generating Triangle, *is an Isoscele, an* obtuse angled Cone; *if the immoveable Side AC be less than the other CD;* and an Accute-angled Cone, *if the immoveable Side AC be greater than the other CD, as it happens in this Figure.* Fig. 4.

A Solid produc'd by the Motion of an oblique angled Triangle, that is to say, one that has not a Right-Angle, is also call'd a Prism. And then to distinguish this Cone from the preceding, 'tis call'd an *Inclined Cone,* as GHI, which is produced by the Motion of the oblique angled Triangle GCH, upon the immoveable Side GO. Fig. 5.

VII.

The Axe of a Cone is the immoveable Side of the generating Triangle: *As AC, passing thro' the Center C of its* Base, and perpendicular to it when it is right. Fig. 4.

VIII.

A *Cylinder* is a Solid bounded by three Surfaces generated by the intire Revolution of a right-angled Parallelogram about one of the Sides that form the Right-Angle.

Thus if you imagine the right-angled Parallelogram GOBC, to revolve about the immoveable Side GO, till the Revolution be intire, that is, till the Side OB, arrive at the Place where it began; the Parallelogram BCGO, will describe by that intire Revolution the Cylinder ABCD. Fig. 6.

A Solid generated by the Motion of a Parallelogram, that has never an Angle right, is also call'd a Cylinder; but then to distinguish it from the foregoing, call'd a *Right Cylinder*, this is call'd *an inclined Cylinder,* as HIKL, which is generated by the Motion of an oblique angled Parallelogram KLNM, about the immoveable Side MN. Fig. 7.

IX.

Fig. 6. *The Axe of a Cylinder* is the immoveable Side of the Parallelogram that generates the Cylinder: *As GF, which is perpendicular to its two Bases, if the Cylinder be a right one.*

X.

Fig. 4. *The Base of a Cone*, is a Circle generated by the Motion of the moveable Side of the generating Triangle. *As BED whose Center is C, thro' which the Axe AC passes.*

XI.

Fig. 6. *The Bases of a Cylinder*, are the two opposite equal and parallel Circles, generated by the Motion of the two opposite equal and parallel Sides of the generating Parallelogram. *As DEC, AFB, whose Centers are G, O, thro' which the Ax GF passes.*

XII.

Similar Cones and Cylinders are such as have their Axes proportional to the Diameters of their Bases.

This Definition belongs to right Cones and Cylinders, for in inclin'd ones, you must add, and their Axes similarly inclin'd to their Bases.

PROPOSITION I.

THEOREM I.

Similar Polygons inscrib'd in Circles are in the same Ratio that the Squares of the Diameters of the Circles are in.

Fig. 8. I Say, if the Polygons AEFBG, CHIDK, inscribed in the Circles, whose Centers are L, M, be similar, they are in the same Ratio as the Squares of the Diameters FN, IO.

PRE-

PREPARATION.

Draw from the two equal Angles F, I, thro' the Centers L, M, the Diameters FN, IO, and from the two other equal Angles E, H, thro' the Extremities N, O, of these Diameters, draw the Right-Lines EN, HO, then draw the Right-Lines AF, CI. Plate 1. Fig. 8.

DEMONSTRATION.

Because the Angles AEF, CHI, are equal, by *Sup.* and the Ratio of the two Sides AE, EF, is equal to that of CH, HI, the Polygons being similar, the two Triangles AEF, CHI, will be similar, by *6. 6.* and the two Angles EAF, HCI, equal, which being also equal to ENF, HOI, by 21. 3. ENF, and HOI are equal, and by 32. 1. the two Triangles NEF, OHI, that are right-angled by 31. 3. being equiangular: Consequently by 4. 6. the four Lines EF, HI, FN, IO, are proportional, and by 22. 6. the Polygon AEFBG form'd upon the first Line EF, is to the similar Polygon CHIDK, form'd upon the second Line HI, as the Square of the third FN is to the Square of the fourth IO. *Which was to be demonstrated.*

USE.

This Proposition serves as a Lemma to the next, and to demonstrate *Prop.* 12. And since we have demonstrated in similar right-angled Triangles NEF, OHI, that the Ratio of the Side EF, to the homologous Side HI, is equal to the Ratio of the Diameter FN, to the Diameter IO, it follows by reason of the Similitude of the Poligons, that the Side AE, is to its homologous Side CH, as the Diameter FN, to the Diameter IO, and so of the other Sides. Whence 'tis easy to conclude by 12. 5. that the Perimiter of the Polygon of the Circle AB, is to the Perimiter of the similar Polygon of the Circle CD, as the Diameter FN is to the Diameter IO. Since the

Fig. 8. greater Number of Sides the Polygon inscribed has, the nearer its Perimeter approaches to the Circumference of the Circle; so that it becomes the Circumference of the Circle, when the Number of Sides of the Polygon is infinite, 'tis evident the Circumference of the Circle AB, is to its Diameter FN, as the Circumference of the Circle CD is to its Diameter IO. And this serves to find the Circumference of a Circle by its Diameter, or the Diameter of a Circle by its Circumference, if we could but once know the Ratio of the Circumference of a Circle to its Diameter, which is as 314 to 100 nearly, as shall be shown in our *Practical Geometry*.

PROPOSITION. II.

THEOREM II.

The Surfaces of Circles are as the Squares of their Diameters.

Fig. 8. I Say the Area of the Circle AB, is to the Area of the Circle CD, as the Square of the Diameter FN is to the Square of the Diameter IO.

DEMONSTRATION.

Because by *Prop.* 1. a Polygon inscrib'd in the Circle AB, is to the similar Polygon inscrib'd in the Circle CD, as the Square of the Diameter FN, is to the Square of the Diameter IO, and this Theorem is generally true of all Polygons, which become Circles, if the Sides be regular and the Number infinite; from whence it follows that the Circles AB, CD, are as the Squares of their Diameters FN, IO. *Which was to be demonstrated.*

COROLLARY I.

Circles are in the Duplicate Ratio of that of their Diameters, because the Squares of their Diameters are in the Duplicate Ratio of that of their Sides, which are the Diameters themselves.

COROL-

COROLLARY II.

Circles are in the ſame Ratio as ſimilar Polygons inſcrib'd, becauſe both of them are as the Squares of the Diameters of the Circles.

USE.

This Propoſition ſerves to find the Area of a Circle, its Diameter being given, if the Ratio of the Area of a Circle to the Square of its Diameter be once known, tho' it is as 785 to 1000 nearly, as ſhall be ſhewn in our *Practical Geometry.*

Prop. III. and IV. are needleſs, *becauſe they only ſerve to demonſtrate* Prop. V. and VI. *that we ſhall demonſtrate otherwiſe and more eaſily, by the Geometry of Indiviſibles.*

PROPOSITION V. and VI.

THEOREM V. and VI.

Pyramids of the ſame Height are as their Baſes.

PYramids of the ſame Height are as their Baſes, whether they be Triangular, as *Prop.* V. requires, or Polygonal, as *Prop.* VI. Becauſe if you imagine Planes parallel to the Baſe, to paſs thro' all the Points of each Height ſuppoſed equal, they will divide each Pyramid into an equal Number of Planes ſimilar to their Baſe, conſequently the Ratio of a Plane of one Pyramid to its Baſe, is the ſame with that of the correſponding Plane of the other Pyramid to its Baſe, by 22. 6. becauſe the Planes and Baſes have their Sides proportional, the ſame Plane cutting their Heights proportionally. Conſequently by 12. 5. all the ſimilar Planes, that make up

one Pyramid are, that is, the whole Pyramid is to its Base, just as many similar Planes that compose the other Pyramid, that is all that Pyramid, is to its Base. *Which was to be demonstrated.*

USE.

This Proposition serves to demonstrate the next, that supposes Pyramids of equal Bases and Heights to be equal, which plainly follows from what has been demonstrated.

PROPOSITION VII.

THEOREM VII.

A Pyramid is the third Part of a Prism of the same Base and Altitude.

I Say first, a Pyramid having for its Base one of the two Triangles BCD, AEF, that are the two parallel similar and equal Bases of the Triangular Prism ABCDEF, and that is of the same Height with the Prism, for instance the Pyramid ABCD, will be the third Part of the same Prism.

DEMONSTRATION.

Draw the three Diagonals AC, AD, CE, and they will divide their Parallelograms into two equal Parts, by 34. 1. the Prism ABCDEF is made up of the three equal Triangular Prisms ABCD, ACDE, ACEF; for the two first, ABCD, ACDE, having the same Vertex C, and consequently the same Height, and their Bases ADB, ADE, equal by 35. 1. are equal, by *Prop.* 5. After the same manner the two last Pyramids ACDE, ACEF, may be found to be equal, because they have the same Vertex A, and consequently the same Height, and their Bases CED, CEF, are equal. Whence it follows that the three Pyramids are equal, and consequently the Pyramid ABCD is the third Part of the Triangular Prism ABCDEF.

ABCDEF, of the same Base and Altitude. *Which was to be demonstrated.*

I say in the second Place, a Pyramid, having its Base of any other Figure, is still the third Part of a Polygonal Prism of the same Base and Altitude, because the Polygonal Prism may be divided into Triangular Prisms, and by that means the Pyramid also will be divided into as many Triangular Pyramids, each of which will be the third Part of its Prism. Consequently by 12. 5. the Polygonal Pyramid is also the third Part of its Polygonal Prism. *Which remain'd to be demonstrated.*

USE.

This Proposition serves to demonstrate the following ones, and find the Solidity of a Pyramid, the Base and Height being given: for since by multiplying the Base of a Pyramid by its Height, you find the Solidity of a Prism, triple the Pyramid, take the third Part of this Solidity, which is the same thing as multiplying the Base by a third Part of its Height, or the Height by the third Part of the Base, and you will have the Solidity of the Prism proposed.

PROPOSITION VIII.

THEOREM VIII.

Similar Pyramids are in the Triplicate Ratio of that of their Homologous Sides.

THis Proposition will be evident, if we imagine upon the Bases of the Pyramids, Similar Prisms of the same Height, which being in the Triplicate Ratio of that of their Homologous Sides, by 33. 11. the similar Pyramids that are their third Parts, by *Prop.* 7. will also be in the triplicate Ratio of that of their Homologous Sides. *Which was to be demonstrated.*

PRO-

PROPOSITION IX.

THEOREM IX.

Equal Pyramids have their Bases and Heights reciprocal: and such as have their Bases and Heights reciprocal, are equal.

I Say first, if two Pyramids are equal, the Base of the first is to the Base of the second, as the Height of the second is to the Height of the first.

DEMONSTRATION.

Imagine upon the Bases of the two Pyramids, Prisms of the same Height, and they will be equal, because by *Prop.* 7. they are triple the Pyramids, that are equal by *Sup.* Consequently by 34. 11. the Bases and Heights of these Prisms, being the same with those of the Pyramids, are reciprocal. *Which was to be demonstrated.*

I say in the second Place, if the Bases and Heights are reciprocal, that is to say, the Base of the first Pyramid to the Base of the second, reciprocally as the Height of the second is to the Height of the first, the two Pyramids are equal.

DEMONSTRATION.

Imagine as before, upon the Bases of the two Pyramids, Prisms of the same Height, by 34. 11. they will be equal, because their Bases and Heights are reciprocal, by *Sup.* Consequently the Pyramids, which are third Parts of them, by *Prop.* 7. are equal. *Which remain'd to be demonstrated.*

PRO-

PROPOSITION X.

THEOREM X.

A Cone is the third Part of a Cylinder of the same Base and Height.

THis Proposition will be evident, if we consider that a Cone is a Pyramid of an infinite Number of Sides, and in like manner, a Cylinder is a Prism of an infinite Number of Sides; and since a Pyramid is the third of a Prism of the same Base and Height, a Cone must also be the third part of a Cylinder of the same Base and Height. *Which was to be demonstrated.*

PROPOSITION XI.

THEOREM XI.

Cylinders and Cones of the same Height, are as their Bases.

THis Proposition will be evident, if we consider that the Bases of Cylinders and Cones being Circles, that is, Regular Polygons of an infinite Number of Sides; Cylinders are Prisms of an infinite Number of Sides, and Cones are Pyramids of an infinite Number of Sides. Consequently what has been said of Prisms in 32. 11. *Prop.* 5. and 6. may be understood of Cylinders and Cones.

PROPOSITION XII.

THEOREM XII.

Similar Cylinders and Cones, are in the Triplicate Ratio of that of the Diameters of their Bases.

I Say first, Similar Cylinders are in the Triplicate Ratio of that of the Diameters of their Bases that are Circles.

DE-

DEMONSTRATION.

Consider a Cylinder as a Parallelopiped, or a Prism of an infinite Number of Sides, and a Circle as a Regular Polygon of an infinite Number of Sides, and by 33. 11. Similar Cylinders are in the Triplicate Ratio of that of their Homologous Sides, and consequently of that of the Diameters of their Bases, that are in the same Ratio as the Homologous Sides of Similar Polygons inscribed in the Bases, by *Prop.* 1. *Which was to be demonstrated.*

I say, in the second place, Similar Cones are also in the Triplicate Ratio of that of the Diameters of their Bases.

DEMONSTRATION.

Consider after the same manner, a Cone as a Pyramid of an infinite Number of Sides, by *Prop.* 8. Cones are in the Triplicate Ratio of that of their Homologous Sides, the same with that of the Diameters of their Bases, by *Prop.* 1. and consequently the Cones are in the Triplicate Ratio of that of the Diameters of their Bases. *Which remain'd to be demonstrated.*

COROLLARY I.

Similar Cones are in the Triplicate Ratio, or as the Cubes of their Axes, because those Axes are in the same Ratio, as the Diameters of their Bases, by reason of the equal Angles made by the Axes and Diameters, since the Cones are suppos'd similar.

COROLLARY II.

Similar Cones are in the Triplicate Ratio, or as the Cubes of their Sides inclined to their Bases, because these Sides are proportional to the Diameters of the Bases, the Angles that the Sides make with the Diameters, being equal. From whence one may easily conclude, that similar Cylinders and Cones are in the Triplicate

Ratio

Ratio of that of their Heights, that serve to demonstrate *Prop.* 18.

PROPOSITION XIII.

THEOREM XIII.

A Cylinder cut by a Plane parallel to its Base, has the Parts of its Axe in the same Ratio as the Parts of the Cylinder.

I Say, if the Cylinder ABCD, be cut by the Plane EF, parallel to the Base AB, or CD, that cuts the Axe GH at the Point I; the Ratio of the Cylinder ABFE, to the Cylinder EFCD, as the Part HI to the Part IG. Fig. 10.

PREPARATION.

Divide each of the two Parts GI, HI, into two equal Parts at the Points O and R, and cause the Planes PQ, MN, parallel to the Base AB, to pass thro' these middle Points O, R, and they will divide the Cylinder EFCD, into two equal Cylinders EFQP, PQCD, and the Cylinder ABFE into two equal Cylinders ABNM, MNFE, by *Prop.* 11. because their Heights, as well as their Bases are equal.

DEMONSTRATION.

Because by 15. 5. the Cylinder AF, is to its half AN, as the Cylinder EC, is to its half EQ; and the Part HI, to its half HR, as the Part IG to its half IO, the Proportion of the four Cylinders AF, AN, EC, EQ, is similar to that of the four Parts HI, HR, IG, IO, consequently by Alternation by 16. 5. you will find the Proportion of the four Cylinders AF, EC, AN, EQ, is similar to that of the four Parts HI, IG, HR, IO, and consequently in this second Proportion, the Ratio of the first Cylinder AF, to the second EC, is equal to that of the first Part HI, to the second IG. *Which was to be demonstrated.*

SCHO

SCHOLIUM.

Fig. 10. This Demonstration is different from the common one, that supposes the two Parts HI, IG, have a common Measure, which is too particular, since they might be incommensurable. For the same reason I have demonstrated the first and last Proposition of the sixth Book.

COROLLARY.

Cylinders of equal Bases are as their Heights, which is of use in the next Poposition; for if you let fall from G in the Axe GH, the Right-Line GK, perpendicular to the Plane of the Base AB, which will also be perpendicular to the Plane of the Base EF, and the Lines HK, IL, be made the common Sections of the two Parallel Planes AB, EF, and the Triangular Plane GKH, you will find by 16. 11. that the two common Sections HK, IL, are parallel, and by 2. 6. that the Ratio of HI to IG, that has been demonstrated to be the same as that of the two Cylinders AF, EC, whose Bases AB, EF, are equal, is equal to that of the Height KL to the Height LG.

PROPOSITION XIV

THEOREM XIV.

Cylinders and Cones of the same Base are as their Heights.

Fig. 11. I Say first, the Ratio of the two Cylinders ABCD, EFGH, that I suppose right ones, is equal to that of their Heights AD, EH, if their Bases AB, EF, are equal.

PREPARATION.

Cut off the greatest Height AD, the Part AI equal to the least Height EH, and suppose the Plane IK to pass thro' the Point I, parallel to the Base AB, and by *Prop.* 11. it will cut off the Cylinder AK, equal to the Cylinder EG.

DE-

DEMONSTRATION.

Because the Cylinder AC, is to the Cylinder AK, as the Height AD, is to the Height AI, by *Prop.* 13. and the Cylinder AK is equal to the Cylinder EG, and the Height AI equal to the Height EH, by *Conſt.* the Cylinder AC, will alſo be to the Cylinder EG, as the Height AD to the Height EH. *Which was to be demonſtrated.* Fig. 11.

I ſay in the ſecond Place, Cones whoſe Baſes are equal, are as their Heights, becauſe they are the third Parts of Cylinders, by *Prop.* 10. whoſe Ratio has been demonſtrated to be equal to that of their Heights.

PROPOSITION XV.

THEOREM XV.

Equal Cylinders and Cones have their Baſes and Heights reciprocal; and ſuch as have their Baſes and Heights reciprocal, are equal.

THis Propoſition is plain from 34. 11. for Cylinders, that are nothing but Parallelopipeds of an infinite Number of Sides, and for Cones by *Prop.* 10. Since they are the third Parts of Cylinders.

I omit Prop. XVI. *and* XVII. *becauſe too perplexing, and only ſerving to demonſtrate the next, that I ſhall demonſtrate a more eaſy way*

PROPOSITION XVIII.

THEOREM XVIII.

Spheres are in the Triplicate Ratio of that of their Diameters.

THis Propoſition will be evident, if we conſider a Sphere is compoſed of an infinite Number of little equal Cones, whoſe common Vertex is the Center of the

the Center of the Sphere, and Height the Radius of the same Sphere, and whose Bases being infinitely small, may pass for Planes and are in the Surface of the Sphere; and consequently the Sum of all these Cones of the same Height, that is, the Solidity of the Sphere is equal, to one Cone whose Height is the same Radius of the Sphere and Base, the intire Surface of the Sphere; and since the Cone equal to this Sphere is similar to a Cone equal to another Sphere, because all Spheres are similar, and similar Cones are in the Triplicate Ratio of their Heights, that here are the Radius's of the two Spheres to which they are equal, it follows that the two Spheres also are in the Triplicate Ratio of their Radii, or Semidiameters, and consequently of their Diameters. *Which was to be demonstrated.*

COROLLARY.

Spheres are as the Cubes of their Diameters, because Cubes are similar Solids, that by 31. 11. are in the Triplicate Ratio of their Sides.

USE.

This Proposition serves to find the Solidity of a Sphere, its Diameter being given; were the Ratio of a Sphere to the Cube of its Diameter but once known, tho' it is as 157 to 300 nearly, as shall be shewn in the *Geometry.*

FINIS.